METRIC PROPERTIES OF HARMONIC FUNCTIONS

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METRIC PROPERTIES OF HARMONIC FUNCTIONS

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In this note we shall formulate two theorems on harmonic functions of two variables. An imprecise, but brief, formulation of Theorem 1 is: *harmonic functions have few long level lines.*

The exact formulation and the outline of the proof are given in § 1. In Theorem 2 a bounded harmonic function that is small on a set of sufficiently large measure is estimated. This theorem is a generalization of Hadamard' s theorem on three circles. Both theorems admit a generalization to the case of elliptic equations with variable coefficients. Theorem 1 is readily carried over to the multidimensional case. Theorem 2 (both in the plane and in space) can also be obtained by a completely different method as a consequence of the results of M. M. Lavrent' ev ([4]).

Throughout what follows, K_R denotes the disk $x^2 + y^2 < R^2$.

§ 1. **On the length of a level line.** Let $u(x, y)$ be a bounded function harmonic in K_R. The set of points $(x, y) \in K_{R/2}$ at which $u(x, y) = t$ will be denoted by E_t. For any t, the set E_t is the union of a finite or countable number of pairwise nonintersecting analytic simple arcs. We denote the sum of the lengths of these arcs by $l(t)$. Let n be a natural number. The set of those t for which $l(t) > 16Rn^2$ will be denoted by $B_{u,n}$. Let U_M be the set of all functions u harmonic in K_R and bounded in absolute value by the constant M.

Theorem 1. *For every $M > 0$ there exists an N such that, for any $n > N$ and any function $u \in U_M$, the measure of $B_{u,n}$ is less than 2^{-n}.*

The example of the function $r^n \sin n\varphi$ in the disk K_1 shows that this estimate cannot be greatly improved. The method of proof of Theorem 1 is close to that used in ([1]).

We outline the proof. Let $u \in U_M$. Fix $t \in B_{u,n}$ so that E_t contains no zeros of $\operatorname{grad} u$. Let $(x, y) \in E_t$. If the tangent to E_t at the point (x, y) forms with the x-axis an angle not exceeding $\pi/4$, assign the point (x, y) to the set X. The set $E_t \setminus X$ will be denoted by Y.

From X we select a subset X_1 by means of the following condition. A point $(x, y) \in X$ is assigned to X_1 if there exists a segment of length $R/2n$, parallel to the y-axis, containing the point (x, y) and intersecting E_t in at least n points. Similarly, if some segment of length $R/2n$, parallel to the x-axis and passing through a point $(x, y) \in Y$, intersects E_t in at least n points, then we assign (x, y) to the set Y_1.

The proof of Theorem 1 is carried out in three stages:

1. It is established that at the points of the set $X_1 \cup Y_1$, for $n > N(M)$, the inequality

$$|\operatorname{grad} u| < \frac{2^{-n}}{R}$$

holds.

2. It is verified that the linear measure of $X_1 \cup Y_1$ is greater than $l(t)/2$.

3. From 1 and 2 it follows that

$$\operatorname{mes} B_{u,n} < 2^{-n}.$$

§ 2. **A generalization of the three-circles theorem.** One of the possible formulations of the theorem on three circles for harmonic functions is as follows*:

* This theorem can easily be obtained from the well-known Hadamard theorem on three circles for analytic functions of a complex variable ([2]).

If $u(x, y)$ is a function harmonic in K_1, and if $|u| < 1$ in K_1, $|u| < \varepsilon^2$ in K_{r^2}, then $|u| < C\varepsilon$ in K_r (ε and r are arbitrary numbers between 0 and 1/2, C is an absolute constant).

How does this estimate change if, in the formulation of the theorem, the circle K_{r^2} is replaced by an arbitrary set $\mathfrak{M}$ of the same measure ($\operatorname{mes}_2 \mathfrak{M} = \pi r^4$)?

The answer to this question is given by

Theorem 2. There exists an $s > 0$ such that for every set $\mathfrak{M}$ of measure πr^4, situated in K_r, from the inequalities

$$|u| < 1 \text{ in } K_1, \qquad |u| < \varepsilon^s \text{ on } \mathfrak{M}$$

it follows that

$$|u| < \varepsilon \text{ in } K_r.$$

(As above, $\Delta u = 0$ in K_1, $0 < \varepsilon, r < 1/2$.)

Let us note that it is enough to prove Theorem 2 for $r = r_0$, in order to be sure of its validity for $r = r_0^n$ (where n is any natural number). The case $r = r_0^n$ is reduced to the case $r = r_0$ if one maps the circle K_1 onto itself by means of the transformation $\sqrt[n]{z}$ $(z = x + iy)$.

Consequently, it is enough to establish that the following is true.

Theorem 2′. There exists an $s > 0$ such that for every set $\mathfrak{M}$ of measure $\pi/16$, situated in $K_{1/2}$, for every function $u(x, y)$ harmonic in K_1, and for every sufficiently large natural number N, from the inequalities

$$|u| < 1 \text{ in } K_1, \qquad |u| < 2^{-sN} \text{ on } \mathfrak{M}$$

it follows that

$$|u| < 2^{-N} \text{ in } K_{1/2}.$$

We outline the plan of the proof of Theorem 2′.

1. We shall call the following construction the k-division of the circle K_R. Consider k concentric circumferences with center 0 and radii $R/k, 2R/k, \dots, iR/k, \dots, R$. Denote by $H_2, \dots, H_k$ the concentric rings of width R/k enclosed between these circumferences. We divide the ring H_i $(1 < i \le k)$ into $2i - 1$ equal regions $D_{i1}, \dots, D_{i,2i-1}$; to D_{ij} belong those points (r, φ) of the ring H_i for which

$$\frac{2\pi}{2i-1}(j-1) < \varphi < \frac{2\pi}{2i-1} j, \qquad 1 \le j \le 2i - 1.$$

   All the regions D_{ij}, obviously, are of equal area: $\operatorname{mes}_2 D_{ij} = \pi R^2/k^2$. We shall call $H_2, \dots, H_k$ the rings of the k-division, and $D_{21}, \dots, D_{k,2k-1}$ the cells of the k-division.

2. By the hypothesis of the theorem, $|u| < 2^{-sN}$ on $\mathfrak{M}$. Denote by $\mathfrak{M}_1$ the set of points of $K_{1/2}$ where $|u| < 2^{1-sN}$. With the help of Theorem 1 it is not difficult to prove that there exists an absolute constant k_0 with the following property:

Let $k \ge k_0(sN)^2$. Carry out the k-division of the circle $K_{1/2}$. Then at least $k^2/8$ cells of the division belong entirely to $\mathfrak{M}_1$.

3. Now, using the theorem on three circles, one can reduce Theorem 2′ to the following lemma:

Lemma 1. Let q and n be natural numbers. Carry out the 2^{q+4}-division of the circle $K_{1/2}$. Let $u(x, y)$ be a harmonic function in the circle K_1, satisfying the conditions: 1) $|u| < 1$ in K_1; 2) among the rings of the division situated between the circumferences of radii $1/32$ and $1/2$, there are $2q$ or more rings containing cells of the division where $|u| < 2^{-2q+n}$ (at least one such cell in each); 3) $\max_{K_{1/2}} |u| > 2^{-2q}$.

Then $n < A$, where A is an absolute constant.

The proof of Lemma 1 recalls the proof of Lemma 1.6.1 in (3).

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Note: Figure translations are in progress. See original paper for figures.