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BOUNDARY-VALUE  
PROBLEMS FOR  
GENERAL PARABOLIC  
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MATHEMATICS

1966

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**Abstract**

**Full Text**

UDC 517.919

**MATHEMATICS**

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## **SCHAUDER ESTIMATES AND SOLVABILITY OF GENERAL BOUNDARY-VALUE PROBLEMS FOR GENERAL PARABOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS**

*(Presented by Academician I. N. Vekua on 9 XI 1965)*

Boundary-value problems for second-order equations with discontinuous coefficients have been studied by many authors (<sup>1-7</sup>). In the present note it is proved that algebraic conditions of the type of the Ya. B. Lopatinskii condition (<sup>8</sup>) on the boundary operators and conjugation operators, and the algebraic conditions on the initial operators recently clarified by V. A. Solonnikov, guarantee the correct solvability of general mixed problems and the Cauchy problem for a broad class of parabolic systems (<sup>14,15</sup>) with discontinuous coefficients in classes of Hölder functions. At the same time, methods developed in the theory of elliptic boundary-value problems with discontinuous coefficients (<sup>9-11</sup>) and parabolic boundary-value problems (<sup>12-17</sup>) are used. We note that, along the way, a priori estimates of Schauder type are obtained for the solutions, and regularizers are constructed for the corresponding boundary-value problems with discontinuous coefficients. In particular, the results obtained include the study of general boundary-value problems with discontinuous coefficients for systems parabolic in the sense of I. G. Petrovskii.

**1°. Notation, formulation of the problem.** Let  $G$  be a bounded domain of the space  $E_n$  with boundary  $\Gamma$ , divided by a surface  $\gamma$  into two subdomains\*  $G_1$  and  $G_2$ , where  $\gamma$  and  $\Gamma$  have no common points, and let  $\Omega_1 = G_1 \times [0, T]$ ,  $\Omega_2 = G_2 \times [0, T]$ ,  $\Omega = G \times [0, T]$  be cylindrical domains of the space  $E_{n+1}$ . Denote by  $S_0 = \gamma \times [0, T]$  and  $S = \Gamma \times [0, T]$ , respectively, the lateral surfaces of the cylinders  $\Omega_1$  and  $\Omega$  ( $0 < T < \infty$ ).

We shall consider in  $\Omega$  linear parabolic, in the sense of V. A. Solonnikov (<sup>14,15</sup>), systems with complex coefficients of the form

$$\mathcal{L} \left( x, t; \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u(x, t) =$$

$$= \begin{cases} \mathcal{L}^{(1)} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u^{(1)}(x, t) = f^{(1)}(x, t), & (x, t) \in \Omega_1, \\ \mathcal{L}^{(2)} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u^{(2)}(x, t) = f^{(2)}(x, t), & (x, t) \in \Omega_2, \end{cases} \quad (1)$$

where  $\mathcal{L}^{(m)} = (l_{ij}^{(m)})$  ( $m = 1, 2$ ) are matrices of dimensions  $N \times N$ . Suppose that there exist integers  $s_i, t_j$  ( $\max_i s_i = 0, (t_j \geq 0, i, j = 1, \dots, N)$ ) such that the degree of the polynomials  $l_{ij}^{(m)}(x, t, i\xi\lambda, \lambda^{2b}p)$  in  $\lambda$  does not exceed  $s_i + t_j$ ,  $b > 0$  an integer. By  $l_{ij}^{(m)0}$  denote the sum of those terms of  $l_{ij}^{(m)}$  satisfying the condition

$$l_{ij}^{(m)0}(x, t, i\xi\lambda, \lambda^{2b}p) = \lambda^{s_i+t_j} l_{ij}^{(m)0}(x, t, i\xi, p), \quad (x, t) \in \Omega_m \quad (m = 1, 2).$$

Let  $\mathcal{L}_0^{(m)} = (l_{ij}^{(m)0})$ . Parabolicity in the sense of V. A. Solonnikov means that the systems (1) have the described structure and that, for any real

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\* All results are valid in the case of a partition of  $G$  into a finite number of subdomains  $G_m$  ( $m = 1, \dots, k$ ).

the vector  $\xi = (\xi_1, \dots, \xi_{n-1}, \tau)$ , the  $p$ -roots of the polynomials

$$L^{(m)}(x, t, i\xi, p) = \det \mathcal{L}_0^{(m)}(x, t, i\xi, p)$$

satisfy the inequality

$$\operatorname{Re} p \leq -\delta \xi^{2b}, \quad \xi^2 = \xi_1^2 + \dots + \xi_{n-1}^2 + \tau^2 = \xi'^2 + \tau^2$$

at every point  $(x, t) \in \Omega_m$  ( $m = 1, 2, \delta > 0$ ) (uniform parabolicity). From the parabolicity condition it follows that, for any point  $(x, t) \in \Omega_m$ ,  $L^{(m)}(x, t, i\xi', i\tau, p)$ , as a polynomial in  $\tau$ , is of even degree,

$$\sum_{i=1}^N (s_i + t_i) = 2br,$$

and the  $\tau$ -roots are distributed equally in the upper and lower half-planes if  $\operatorname{Re} p \geq -\delta_1 \xi'^{2b}$ ,  $|p| + |\xi'| \neq 0$ ,  $\delta_1 < \delta$ .

In what follows, by  $v^{(m)}(x, t)$  we shall mean the value of the function  $v(x, t)$  for  $(x, t) \in \Omega_m$ , and by  $v^{(m)}(x_0, t_0)|_{S_0}$  the limit of  $v^{(m)}(x, t)$  as  $(x, t)$  tends to  $(x_0, t_0) \in S_0$ .

Let us now consider the problem of finding a solution  $u(x, t)$  of system (1) in  $\Omega - S_0$ , satisfying on the surface  $S$  the boundary conditions

$$\sum_{j=1}^N d_{\alpha j} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j(x, t) \Big|_S = \Phi_\alpha(x, t) \quad (\alpha = 1, \dots, br), \quad (2)$$

and, for  $t = 0$ , the initial conditions\*

$$\sum_{j=1}^N c_{hj}^{(m)} \left( x, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j^{(m)}(x, t) \Big|_{t=0} = \varphi_h^{(m)}(x) \quad (h = 1, \dots, r), \quad (3)$$

and on  $S_0$  the conjugation conditions

$$\sum_{j=1}^N \left[ b_{qj}^{(1)} \left( x, t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) u_j^{(1)}(x, t) - b_{qj}^{(2)} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j^{(2)}(x, t) \right] \Big|_{S_0} = \psi_q(x, t) \quad (q = 1, \dots, 2br). \quad (4)$$

It is assumed that there exist integers  $m_\alpha, \rho_h, \sigma_q$  such that  $d_{\alpha j}, c_{hj}^{(m)}, b_{qj}^{(m)}$  are linear differential operators of  $o$ -orders respectively  $m_\alpha + t_j, \rho_h + t_j, \sigma_q + t_j$ . If the order turns out to be negative, the element is equal to zero. By  $\mathcal{D}_0, C_0^{(m)}, \mathcal{B}_0^{(m)}$  ( $m = 1, 2$ ) we denote the matrices composed of the principal parts of the corresponding elements, which are defined analogously to  $l_{ij}^{(m)0}$ .

**2°. Algebraic conditions on the boundary, initial, and conjugation operators.** Let  $l_0 = \max(0, \sigma_q, m_\alpha)$ ,  $t = \max t_j$ ,  $l > l_0$ , where  $l$  is a noninteger, and let the surfaces  $S_0$  and  $S$  be Lyapunov surfaces and belong to the class  $C^{l+t}$ . In this subsection we assume, for the time being, that the coefficients of all operators are sufficiently smooth. Everywhere in what follows it is assumed that: 1) the operator  $\mathcal{D}(x, t, \partial/\partial x, \partial/\partial t)$  satisfies the complementing condition (see (15), p. 11); 2) the rows of the matrices  $C^{(m)}(x, 0, p) \widehat{\mathcal{L}}(x, 0, 0, p)$ , considered as polynomials in  $p$ , are linearly independent modulo  $p^r$ , where  $\widehat{\mathcal{L}}_0^{(m)} = L^{(m)}(\mathcal{L}_0^{(m)})^{-1}$ .

We proceed to the description of the conditions imposed on the conjugation operators. Let  $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$  be the unit vector of the inner normal to  $S_0$  at the point  $x$ , and let  $\zeta(x)$  be any vector lying in the tangent

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\* As shown in (15), the question of the existence and construction of the matrices  $C^{(m)}$  for arbitrary systems of the form (1) is connected with the topological structure of the domains  $G_m$  ( $m = 1, 2$ ). Since the domain  $G_2$  is not homeomorphic to a ball, the question of the existence of  $C^{(2)}$  remains open in the general case, and, in general, the initial conditions may have to be prescribed differently

(15). In the case when system (1) is parabolic in the sense of I. G. Petrovsky, the initial conditions are prescribed in the usual way (the Cauchy data).

plane to  $S_0$  at the point  $x$ . Consider the  $\tau$ -polynomials  $L^{(m)}(x, t, i(\xi + \tau\nu), p)$  and factorize them:

$$L^{(m)}(x, t, i(\xi + \tau\nu), p) = M_+^{(m)}(x, t, \xi, \tau, p)M_-^{(m)}(x, t, \xi, \tau, p) \quad (m = 1, 2),$$

where  $M_+^{(m)}$  ( $M_-^{(m)}$ ) have  $\tau$ -roots with  $\text{Im } \tau > 0$  ( $\text{Im } \tau < 0$ ). Form the matrices polynomial in  $\tau$

$$\mathcal{A}_m(x, t, i(\xi + \tau\nu), p) = \mathcal{B}_0^{(m)}(x, t, i(\xi + \tau\nu), p)\hat{\mathcal{L}}_0^{(m)}(x, t, i(\xi + \tau\nu), p).$$

Denote by  $\mathcal{A}'_1$  and  $\mathcal{A}'_2$  the matrices of the remainders after division with respect to  $\tau$  of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively, by  $M_+^{(1)}$  and  $M_-^{(2)}$ .

**Definition.** The operators  $\mathcal{B}^{(1)}(x, t, \partial/\partial x, \partial/\partial t)$  and  $\mathcal{B}^{(2)}(x, t, \partial/\partial x, \partial/\partial t)$  jointly cover on  $S_0$  the parabolic operator  $\mathcal{L}(x, t, \partial/\partial x, \partial/\partial t)$  with discontinuous coefficients if, for any point  $(x, t) \in S_0$  and any tangential vector  $\xi(x)$ , the rows of the matrix

$$(\mathcal{A}'_1(x, t, i(\xi + \tau\nu), p), \mathcal{A}'_2(x, t, i(\xi + \tau\nu), p))$$

are linearly independent with respect to  $\tau$  for  $\text{Re } p > -\delta_0\xi^{2b}$ ,  $\delta_0 < \delta_1$ .

**3°. Restrictions on the coefficients of the operators.** Compatibility conditions. Let  $l > l_0$ ,  $l$  not an integer. Denote by  $C_{xt}^{l, l/2b}(\Omega_1)$  the class of functions whose derivatives of order  $[l]$  with respect to  $x$  are Hölder continuous with exponent  $l - [l]$ , and whose derivatives of order  $[l/2b]$  with respect to  $t$  are Hölder continuous with exponent  $l/2b - [l/2b]$  in  $\Omega_1$ . The classes  $C_{xt}^{l, l/2b}(\Omega_2)$ ,  $C_{xt}^{l, l/2b}(S_0)$ ,  $C_{xt}^{l, l/2b}(S)$  are defined analogously. The norms in these spaces are introduced in the usual way.

Consider the spaces

$$C_{xt}^{l, l/2b}(\Omega) = C_{xt}^{l, l/2b}(\Omega_1) \dot{+} C_{xt}^{l, l/2b}(\Omega_2), \quad C^l(G) = C^l(G_1) \dot{+} C^l(G_2),$$

in which the norm is defined as the sum of the norms of the components. Suppose that  $S_0, S \in C^{l+t}$ , and that the coefficients of the operators  $l_{ij}$ ,  $d_{\alpha j}$ ,  $c_{hj}$ ,  $b_{qj}^{(m)}$  belong respectively to the classes

$$C_{xt}^{l-s_i, (l-s_i)/2b}(\Omega), \quad C_{xt}^{l-m_\alpha, (l-m_\alpha)/2b}(S),$$

$$C^{l-\rho_h}(G), \quad C_{xt}^{l-\sigma_q, (l-\sigma_q)/2b}(S_0).$$

If condition 2) of item 2° imposed on the matrix of initial conditions is satisfied, then, as shown in <sup>(15)</sup>, from the system and the initial conditions one can determine any a priori existing derivative of any function  $u_j^{(m)}$  ( $m = 1, 2$ ) for  $t = 0$ ,  $x \in G - \gamma$ . We shall say that the compatibility conditions of order  $k$  are satisfied for the boundary-value problem with discontinuous coefficients if the functions  $u_j^{(m)}$  and their derivatives at  $t = 0$ , determined from the system and the initial conditions, satisfy on  $S$  the boundary conditions (see <sup>(15)</sup>, p. 87), and on the surface  $S_0$  the conditions

$$\begin{aligned} & \frac{\partial^{i_q}}{\partial t^{i_q}} \sum_{j=1}^N \left[ b_{qj}^{(1)} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j^{(1)}(x, t) \right. \\ & \left. - b_{qj}^{(2)} \left( x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) u_j^{(2)}(x, t) \right]_{\substack{x=s_0 \\ t=0}} = \frac{\partial^{i_q}}{\partial t^{i_q}} \psi_q \Big|_{t=0} \\ & \left( i_q = 0, 1, \dots, \left[ \frac{k - \sigma_q}{2b} \right]; \quad q = 1, \dots, 2br \right). \end{aligned}$$

The following basic theorem holds.

**Theorem.** Let  $l > l_0$ ,  $l$  not an integer,  $S_0$  and  $S$  be Lyapunov surfaces and belong to  $C^{l+t}$ . Let  $\mathcal{B}^{(1)}$  and  $\mathcal{B}^{(2)}$  jointly cover  $\mathcal{L}$  on  $S_0$ ,

$\mathfrak{D}(x, t, \partial/\partial x, \partial/\partial t)u$ ,  $C^{(m)}(x, \partial/\partial x, \partial/\partial t)u$  satisfy, respectively, conditions 1) and 2) of § 2°, and the coefficients of the operators under consideration belong to the classes indicated above.

If

$$f_i \in C_x^{l-s_i} C_t^{(l-s_i)/2b}(\Omega), \quad \varphi_h \in C^{l-\rho_h}(G),$$

$$\Phi_\alpha \in C_x^{l-m_\alpha} C_t^{(l-m_\alpha)/2b}(S), \quad \psi_q \in C_x^{l-\sigma_q} C_t^{(l-\sigma_q)/2b}(S_0),$$

and the compatibility conditions of order  $[l]$  are satisfied, then problem (1)–(4) has a unique solution  $u = (u_1, \dots, u_N)$  such that

$$u_j \in C_x^{l+t_j} C_t^{(l+t_j)/2b}(\Omega)$$

and for which the inequality

$$\sum_{j=1}^N \|u_j\|_{C_x^{l+t_j} \tau^{(l+t_j)/2b}(\Omega)} \leq c_0 \left( \sum_{i=1}^N \|f_i\|_{C_x^{l-s_i} \tau^{(l-s_i)/2b}(\Omega)} + \sum_{h=1}^r \|\varphi_h\|_{C^{l-\rho_h}(G)} + \sum_{q=1}^{2br} \|\psi_q\|_{C_x^{l-\sigma_q} \tau^{(l-\sigma_q)/2b}(S_0)} + \sum_{\alpha=1}^{br} \|\Phi_\alpha\|_{C_x^{l-m_\alpha} \tau^{(l-m_\alpha)/2b}(S)} \right),$$

where  $c_0$  does not depend on  $f_i, u_j, \varphi_h, \psi_q, \Phi_\alpha$ .

The proof of this theorem is carried out with the aid of the method developed in the theory of elliptic boundary-value problems with discontinuous coefficients<sup>(9-11)</sup> and of the technique set forth in<sup>(12-17)</sup>. In particular, a regularizer is constructed and the correct solvability of mixed problems for systems with discontinuous coefficients is established. Let us note that analogous results are obtained in the case when the domains  $\Omega_1$  and  $\Omega$  are noncylindrical, and also when  $G_2 = \bar{E}_n \setminus G_1$  (the case of the Cauchy problem).

In conclusion I express my sincere gratitude to S. D. Eidelman for posing the problem, supervising the work, and for the assistance rendered.

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Received  
3 IX 1965

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