ON APPROXIMATIONS OF ERGODIC DYNAMICAL SYSTEMS BY PERIODIC TRANSFORMATIONS

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ON APPROXIMATIONS OF ERGODIC DYNAMICAL SYSTEMS BY PERIODIC TRANSFORMATIONS

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1. Let (M, μ) be a Lebesgue space (see ([1])); let ξ_n be a partition of (M, μ) into sets C_n^i of measure $1/n$, $i = 1, \dots, n$; and let $\mathfrak{A}_n$ be the algebra of sets measurable with respect to the partition ξ_n ([1]). We shall call an automorphism S_n of the space (M, μ) **cyclic with respect to the partition** ξ_n, if:

P.1. $S_n \xi_n = \xi_n$.

P.2. $S_n^n = E, \quad S_n^k \neq E$ for $k < n$.

We shall say that an automorphism T of the space (M, μ) **admits approximation by cyclic transformations with speed** $o[f(q_n)]$, if, for an increasing sequence of natural numbers q_n, there exists a sequence of partitions $\xi_{q_n} \to \varepsilon^*$ and a sequence of automorphisms S_{q_n}, cyclic with respect to the partitions ξ_{q_n}, such that

$$\sum_{i=1}^{q_n} \mu(TC_{q_n}^i \,\Delta\, S_{q_n} C_{q_n}^i) = o[f(q_n)].$$

We shall indicate the connection of this concept with the concept of the entropy of an automorphism (for the definition and properties of entropy see, for example, the survey ([2])).

Theorem 1. *If an automorphism T admits approximation by cyclic transformations with speed $o(1/\ln^2 q_n)$, then $h(T) = 0$.*

Proof. It is known that for any k_n

$$0 \leq H(T, \xi_n) \leq H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n} \xi_n)/k_n. \tag{1}$$

Let

$$\mu_n = \sum_{i=1}^{q_n} \mu(TC_{q_n}^i \,\Delta\, S_{q_n} C_{q_n}^i).$$

From the condition of the theorem it follows that $\mu_n = \alpha_n / \ln^2 q_n$ ($\alpha_n \to 0$). From the properties of entropy and the condition of the theorem we have:

$$\frac{H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n}\xi_n)}{k_n} \leq -\frac{\ln(1/q_n) + k_n\mu_n \ln(k_n\mu_n/q_n^{k_n+1})}{k_n}.$$

Put $k_n = [\ln q_n/\sqrt{\alpha_n}] + 1$. Then

$$H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n}\xi_n)/k_n \to 0. \qquad (2)$$

From the results of ([5]) it follows that if $\xi_n \to \varepsilon$, then $H(T, \xi_n) \to h(T)$. From this remark and formulas (1) and (2), Theorem 1 follows.

Remark. Theorem 1 remains valid if one requires that the automorphisms S_{q_n} satisfy only condition P.1.

With the concept of the speed of approximation by periodic transformations one can associate various numerical invariants of dynamical sy-

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* ε is the partition of the space (M, μ) into individual points.

systems based on the idea of the "best" admissible rate of approximation. One of the possible constructions is as follows.

Let $X(T)$ be the set of those α for which the automorphism T admits approximation by cyclic transformations with rate $o(1/q_n^\alpha)$. Then put

$$d(T) = \sup\{\alpha : \alpha \in X(T)\}.$$

It follows from Theorem 1, for example, that if $h(T) > 0$, then $d(T) = 0$. It will be clear from what follows that the invariant $d(T)$ is nontrivial.

2. A higher rate of approximation makes it possible to draw conclusions about the spectral properties of the automorphism T.

Theorem 2. *An automorphism T admitting approximation by cyclic transformations with rate $o(1/q_n)$ is ergodic.*

Theorem 3. *If an automorphism T admits approximation by cyclic transformations with rate $o(1/q_n)$, then there is strong convergence*

$$U_T^{q_n} \Rightarrow E,$$

where U_T is the unitary operator in $L^2(M, \mu)$ associated with the automorphism T:

$$U_T(f(x)) = f(T^{-1}x).$$

Corollaries of Theorem 3. *If an automorphism T is approximated by cyclic transformations with rate $o(1/q_n)$, then:*

1°. *The maximal spectral type of the operator U_T is singular.*

2°. *The automorphism T is not mixing.*

Let now (M', μ') be the direct product of the space (M, μ) with the two-point space $Z_2 = (+1, -1)$ with measures $(1/2, 1/2)$. Let $j \in Z_2$. Let H_1 be the subspace of $L^2(M', \mu')$ consisting of functions $f(x, j)$ for which $f(x, 1) = f(x, -1)$; let H_{-1} be the subspace consisting of functions $f(x, j)$ for which $f(x, 1) = -f(x, -1)$. We shall say that an automorphism T' of the space M' is constructed from an automorphism T of the space M and a function $n(x)$, if

$$T'(x, j) = (T(x),\, n(x)j),$$

where $n(x)$ is a function on M with values ± 1.

Suppose the automorphism T satisfies the conditions of Theorem 1, and the set $N\{x : n(x) = -1\}$ has the following property: there exist sets $N_{q_n} \in \mathfrak{A}_{q_n}$ such that

$$\mu(N_{q_n} \Delta N) = o(1/q_n),$$

and moreover the set N_{q_n} consists of an odd number of elements ξ_{q_n}. Under these assumptions the following is true.

Theorem 3′. *In the subspace H_{-1} there is strong convergence*

$$U_{T'}^{q_n} \Rightarrow -E.$$

For the study of the properties of concrete dynamical systems, we shall need, in addition to the theorems formulated above, the following lemmas.

Lemma 1. *If a normalized measure σ on the unit circle is such that*

$$\int \xi^{q_n}\, d\sigma(\xi) \to \xi_0, \qquad q_n \to \infty, \qquad |\xi_0| = 1, \quad \xi_0 \neq 1,$$

then $\sigma \perp \sigma * \sigma$.

Remark. Lemma 1 remains valid also in the case of real time, in the following formulation:

Let $\sigma(\lambda)$ be a normalized measure on the line such that

$$\int e^{it_n\lambda}\, d\sigma(\lambda) \to \xi_0, \qquad t_n \to +\infty, \qquad |\xi_0| = 1, \quad \xi_0 \neq 1;$$

then $\sigma \perp \sigma * \sigma$.

We shall say that an ordered pair of numbers (A, B) satisfies condition C if:

C.1. $0 < A < 1,\ 0 < B < 1$, and the number A is irrational.

C.2. There exists a sequence p_n/q_n of convergents of the number A such that

$$|p_n/q_n - A| = o(1/q_n^2).$$

C.3. There exists $c > 0$ such that for all integers r

$$|r/q_n - B| > c/q_n.$$

Lemma 2. *If the pair of numbers* (A, B) *satisfies condition C, then the function on the unit circle*

$$g(x) = e^{i\lambda_1} \quad \text{for } 0 \le x < 1 - B; \qquad g(x) = e^{i\lambda_2} \quad \text{for } 1 - B \le x \le 1$$

$(e^{i\lambda_1} \ne e^{i\lambda_2})$, *cannot be represented in the form*

$$h(x + A)/h(x),$$

where $h(x)$ *is a measurable function (all equalities for functions here and below are understood mod* 0*).*

3. We pass to the consideration of examples of applications of our general theorems. We begin with the construction of examples of an ergodic automorphism and an ergodic flow whose maximal spectral types do not dominate their convolution.

Let (M, μ) be the circle with Lebesgue measure, $T_\alpha(x) = x + \alpha$, where α is an irrational number. Consider the automorphism $T' = T_{\alpha,\gamma}$ of the space $M' = M \times Z_2$, constructed from the automorphism T_α and the function $n(x) = -1$ for $0 \le x < \gamma$, $n(x) = 1$ for $\gamma \le x \le 1$. Applied to $T_{\alpha,\gamma}$, the theorem and the lemma of § 2 give:

3.1. *If the pair* (α, γ) *satisfies condition C, then the spectrum of the operator* $U_{\alpha,\gamma}$ *in the invariant subspace* H_{-1} *is continuous (Lemma 2).*

3.2. *If there exist sequences of rational numbers* k_n/m_n *and* l_n/m_n $(m_n \to \infty)$ *such that*

$$|\alpha - k_n/m_n| = o(1/m_n^2); \qquad |\gamma - l_n/m_n| = o(1/m_n);$$

$$\text{the numbers } l_n \text{ are odd,} \tag{*}$$

then the conditions for the applicability of Theorem $3'$ *and Lemma 1 are fulfilled.*

Thus, if the pair (α, γ) satisfies both condition C and condition $(*)$, then the maximal spectral type σ of the operator $U_{\alpha,\gamma}$, conjugate to $T_{\alpha,\gamma}$, is represented in the form $\sigma_1 + \sigma_{-1}$, where σ_1 is the discrete type concentrated at the points $\{e^{2\pi i n\alpha}\}$; σ_{-1} is a continuous type, $\sigma_{-1} * \sigma_{-1} \perp \sigma_{-1}$.

Consequently, σ does not dominate $\sigma * \sigma$.

3.3. *If the numbers α and γ are rationally independent, then the multiplicity of the spectrum of the operator $U_{\alpha,\gamma}$ does not exceed 2, since the shifts of the functions $N(x,j) = jn(x)$ and $M(x,j) = jm(x)$, where $m(x) = 1$ for $0 \le x < \alpha$, $m(x) = 0$ for $\alpha \le x \le 1$, generate the whole subspace H_{-1}.*

4. Let now (M, μ) be the unit torus with Lebesgue measure; x, y are cyclic coordinates on the torus; $T_t^{(\alpha)}(x,y) = T(x+t, y+\alpha t)$; α is an irrational number.

Define automorphisms $T_t^{(\alpha,\gamma)}$ of the space $M \times Z_2$ for $0 \le t \le 1$ by the formula

$$T_t^{(\alpha,\gamma)}(x,y,j) = (T_t^{(\alpha)}(x,y),\, jn_t(x,y)),$$

where $n_t(x,y) = -1$ for $1-t \le x \le 1$, $1-\alpha-\gamma \le y - \alpha x \le 1-\alpha$; $n_t(x,y) = 1$ for the remaining x, y. For integers n set $T_n^{(\alpha,\gamma)} = [T_1^{(\alpha,\gamma)}]^n$. Let now $t = n + \tau$, where $0 \le \tau < 1$. Set

$$T_t^{(\alpha,\gamma)} = T_n^{(\alpha,\gamma)} \cdot T_\tau^{(\alpha,\gamma)}.$$

It is easy to verify that the transformations $T_t^{(\alpha,\gamma)}$ defined in this way form a group.

Let $U_t^{(\alpha,\gamma)}$ be the group of unitary operators in $L^2(M \times Z_2)$ conjugate to the flow $T_t^{(\alpha,\gamma)}$. Applying the results of § 2 to the automorphism $T_1^{(\alpha,\gamma)}$ and taking into account that the flow $T_t^{(\alpha,\gamma)}$ is ergodic, we obtain:

4.1. *If the pair of numbers (α,γ) satisfies condition C, then the spectrum of the group $U_t^{(\alpha,\gamma)}$ in the subspace H_{-1} is continuous.*

4.2. *If the pair of numbers (α,γ) satisfies condition $(*)$, then in the subspace H_{-1} there is strong convergence*

$$U_{m_n}^{(\alpha,\gamma)} \Rightarrow -E.$$

Taking into account the remark to Lemma 1, we obtain that, when conditions C and $(*)$ are simultaneously fulfilled, *the maximal spectral type of the group $U_t^{(\alpha,\gamma)}$ does not dominate its convolution.*

4.3. We shall assume that the coordinates x, y on the torus are chosen so that the conditions are fulfilled:

$$0 \le y - \alpha x < 1, \qquad 0 \le x < 1.$$

Consider the subspace $H^{(n)}$ of functions constant on the intervals

$$\delta_{k,\beta}: \ \{y - \alpha x = \beta,\ k/2^n \le x < (k+1)/2^n\}, \quad 0 \le \beta < 1, \quad k = 0, \dots, 2^n - 1.$$

This subspace is invariant with respect to the operator $U_{2^{-n}}^{(\alpha,\gamma)}$, and the operator $U_{2^{-n}}^{(\alpha,\gamma)}$, restricted to this subspace, is spectrally isomorphic to the operator $U_{\alpha/2^n,\gamma/2^n}$ of item 3. Thus, if the numbers α and γ are rationally independent, then the multiplicity of the spectrum of the group $U_t^{(\alpha,\gamma)}$ in the invariant subspaces spanned by $H^{(n)}$ does not exceed two. Since $H^{(n)} \nearrow$

$L^2(M \times Z_2)$, whence it follows that the maximal spectral multiplicity of the group $U_t^{(\alpha,\gamma)}$ does not exceed two.

5. As another application of our methods, let us study the following transformation of the interval $[0, 1]$.

Let $0 < \alpha < \beta < 1$. Put $P_{\alpha,\beta}(x) = x+1-\alpha$ for $0 \le x < \alpha$; $P_{\alpha,\beta}(x) = x+1-\alpha-\beta$ for $\alpha \le x < \beta$; $P_{\alpha,\beta}(x) = x - \beta$ for $\beta \le x < 1$. This transformation mod0 is evidently the rearrangement of the intervals $[0, \alpha]$, $[\alpha, \beta]$, $[\beta, 1]$ in reverse order. Let $f(x) \in L^2([0, 1])$. Put, as usual, $V_{\alpha,\beta}(f(x)) = f(P_{\alpha,\beta}^{-1}(x))$.

Let C be the circle $0 \le \varphi < 1 + \beta - \alpha$; $T(\varphi) = \varphi + 1 - \alpha$. The derived transformation ([4]), induced by the shift T on the arc $l = [0, 1]$, is isomorphic to the transformation $P_{\alpha,\beta}$. It follows that the transformation $P_{\alpha,\beta}$ is ergodic or nonergodic simultaneously with the shift T, and, consequently, a necessary and sufficient condition for the ergodicity of the transformation $P_{\alpha,\beta}$ is the irrationality of the number $(1 - \alpha)/(1 + \beta - \alpha)$ or $\beta/(1 - \alpha)$. Denote

$$(1 - \alpha)/(1 + \beta - \alpha) = A, \qquad (\beta - \alpha)/(1 + \beta - \alpha) = B.$$

Apply to the transformation the results of §2.

5.1. If the pair of numbers (A, B) satisfies condition C, then the spectrum of the operator $V_{\alpha,\beta}$ is continuous.

Let us demonstrate on this example the reduction of such a theorem to Lemma 2.

Let $f(x)$ be an eigenfunction of the operator $V_{\alpha,\beta}^{-1}$, i.e.

$$V_{\alpha,\beta}^{-1}(f(x)) = e^{i\lambda} f(x)$$

($f(x) \ne \text{const}$, $e^{i\lambda} \ne 1$). Define a function $f^*(\varphi)$ on the circle C: $f^*(\varphi) = f(\varphi)$ for $0 \le \varphi < 1$; $f^*(\varphi) = f(\varphi + 1 - \alpha)$ for $1 \le \varphi < 1 + \beta - \alpha$. The function

$$\hat{f}(z) = f^*(z(1 + \beta - \alpha))$$

is defined on the circle $0 \le z < 1$, and

$$\hat{f}(z + A)/\hat{f}(z) = g(z) = 1$$

for $1 - \beta \le z < 1$;

$$g(z) = e^{i\lambda}$$

for $0 \leq z < 1-\beta$. Thus, if the function $g(z)$ cannot be represented in the form $h(z+A)/h(z)$, then the operator $V_{\alpha,\beta}$ has no eigenfunctions.

5.2. If the pair of numbers (A, B) satisfies condition $(*)$, and the numbers l_n are arbitrary, then

$$V_{\alpha,\beta}^{m_n - l_n} \to E'u,$$

and consequently the spectrum of the operator $V_{\alpha,\beta}$ is singular, and the transformation $P_{\alpha,\beta}$ is not mixing.

5.3. From the theorem proved by V. I. Oseledets ([2]), it follows that the spectral multiplicity of the automorphism $P_{\alpha,\beta}$ in the ergodic case does not exceed two.

Theorem 4. There exists a continuum of pairs of numbers (α, β) for which the spectrum of the operator $V_{\alpha,\beta}$ is simple.

Among this continuum of pairs (α, β) there exist some such that, for the pair

$$A = (1-\alpha)/(1+\beta-\alpha)$$

and

$$B = (\beta-\alpha)/(1+\beta-\alpha),$$

condition C is fulfilled. Consequently, among the automorphisms $P_{\alpha,\beta}$ there exist automorphisms with simple continuous spectrum.

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Note: Figure translations are in progress. See original paper for figures.