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ON THE PROPAGATION OF HIGH-FREQUENCY SIGNALS IN A MEDIUM WITH RANDOM CHARACTERISTICS

We consider an equation for the propagation of a signal Ψ of the form

$$A_{jk}\frac{\partial^2\Psi}{\partial x_j\partial x_k}+B_k\frac{\partial\Psi}{\partial x_k}+C\Psi=0, \tag{1}$$

where the coefficients A_{jk}, B_k, C are random functions of the variables x_j $(j=1,2,3,4)$. It is assumed that, in the range of possible values of A_{jk}, equation (1) remains hyperbolic. Next put

$$A_{jk}=\overline{A}_{jk}+a_{jk},\qquad B_k=\overline{B}_k+b_k,\qquad C=\overline{C}+c, \tag{2}$$

where the bar denotes averaging over possible values of the random quantities A_{jk}, B_k, C. This averaging has the meaning of functional integration over possible values of the random quantity $a(x)$:

$$\overline{\Phi}=\int\Phi\{a(x)\}\,dw\{a(x)\}, \tag{3}$$

where Φ is a functional of $a(x)$; $dw\{a(x)\}$ is the probability that $a=a(x)$. We shall assume that the random quantity $a(x)$ can be represented in the form of a series

$$a(x)=\sum_n a_n\varphi_n(x,\alpha_n), \tag{4}$$

where $\varphi_n(x,\alpha_n)$ is some system of orthonormal functions, α_n are random phases, and a_n are random amplitudes. In view of (4), $dw\{a(x)\}$ may be regarded as the probability of one or another set of values of the quantities a_n, α_n; in particular, if a_n, α_n are independent, then

$$dw\{a(x)\} = \prod_n dw(a_n)\, d\Omega(\alpha_n). \tag{5}$$

We shall seek the solution Ψ in the form

$$\Psi = Ae^{iS}, \tag{6}$$

where the frequency ω considerably exceeds the frequencies characteristic of the spectrum of the random quantities A_{jk}, B_k, C. In this case the amplitude A and the phase function S may be regarded as slowly varying functions of the variables x_j (the geometrical-optics approximation). Setting $S = \overline{S} + \sigma$ and substituting (6) into (1), as $\omega \to \infty$ we obtain:

$$\overline{A}_{jk}\frac{\partial\overline{S}}{\partial x_j}\frac{\partial\overline{S}}{\partial x_k} = 0, \tag{7}$$

$$\overline{A}_{jk}\frac{\partial\overline{S}}{\partial x_j}\frac{\partial\sigma}{\partial x_k} + \overline{A}_{jk}\frac{\partial\sigma}{\partial x_j}\frac{\partial\overline{S}}{\partial x_k} + a_{jk}\frac{\partial\overline{S}}{\partial x_j}\frac{\partial\overline{S}}{\partial x_k} = 0. \tag{8}$$

From the last equation one finds the random phase σ, which will be a linear functional of the random quantities $a_{jk}(x)$. Therefore $\sigma(x)$ has the form

$$\sigma(x) = \sum_n a_n \sigma_n(x, \alpha_n), \tag{9}$$

where $\sigma_n(x, a_n)$ corresponds to the solution of the system (7), (8), if in (4) all $a_m = 0$ are set, except for a_n.

The mean value of the signal Ψ will be

$$\overline{\Psi} = Ae^{i\omega\overline{S}}\overline{e^{i\omega\sigma}}. \tag{10}$$

If the distribution $dw(a_n)$ is normal,

$$dw(a_n) = \frac{1}{\sqrt{\pi}}\exp\left[-\frac{(a_n - \overline{a}_n)^2}{b_n^2}\right]\frac{da_n}{b_n}, \tag{11}$$

then, on the basis of (9), we obtain

$$\overline{\Psi} = Ae^{i\omega\overline{S}}\prod_n\int_0^{2\pi} d\Omega(\alpha_n)\exp\left[i\omega\overline{a}_n\sigma_n(x,\alpha_n) - \frac{b_n^2\omega^2}{5}\sigma_n^2(x,\alpha_n)\right]. \tag{12}$$

The final result depends on the form of $\sigma(x, \alpha_n)$.

Let us consider several applications.

A. Scattering of sound in a turbulent flow. In the simplest case of a stationary, vortex-free flow, neglecting quantities of order u^2/c^2 (u is the flow velocity, c is the speed of sound), the equation for the velocity potential of the sound wave φ is [(1)]

$$\partial^2\varphi/\partial t^2 + 2u\,\partial^2\varphi/\partial x\,\partial t - c^2\partial^2\varphi/\partial x^2 = 0, \tag{13}$$

so that $x_1 = x,\ x_4 = t$ and $A_{44} = 1,\ A_{41} = a_{41} = 2u,\ A_{11} = -c^2$.

Equations (7) and (8) now have the form

$$\left(\frac{\partial \overline{S}}{\partial t}\right)^2 - c^2\left(\frac{\partial \overline{S}}{\partial x}\right)^2 = 0, \tag{7'}$$

$$\frac{\partial \overline{S}}{\partial t}\frac{\partial \sigma}{\partial t} - 2c^2\frac{\partial \overline{S}}{\partial x}\frac{\partial \sigma}{\partial x} + 2u\frac{\partial \overline{S}}{\partial x}\frac{\partial \overline{S}}{\partial t} = 0. \tag{8'}$$

From (7) for a plane wave we have $\overline{S} = t \pm x/c$, so that

$$\partial\sigma/\partial t \mp 2c\,\partial\sigma/\partial x + 2u/c = 0. \tag{14}$$

For a stationary flow $\partial u/\partial t = 0$, and one may obtain $\partial\sigma/\partial t = 0$. Then we obtain

$$\sigma(x) = \pm\int_0^x \frac{u(x')\,dx'}{c^2}. \tag{15}$$

Now putting

$$u(x) = \sum_n u_n \cos(q_{nx} + \alpha_n), \tag{16}$$

for the normal distribution law of u_n, with $\overline{u}_n = 0$, we obtain

$$\overline{e^{i\omega\sigma}} = \prod_n \int \exp\left[-\frac{\omega^2 b_n^2}{4q^2c^4}F_n^2(x, \alpha_n)\right]\frac{d\alpha_n}{2\pi}, \tag{17}$$

where

$$F_n(x, \alpha_n) = \sin(q_{nx} + \alpha_n). \tag{18}$$

Since $F^2(x, \alpha_n) > 0$ and is bounded, this quantity in the exponent of (17) may be replaced by the effective mean $F_n^2(x, \theta_n \alpha_n)$, $0 < \theta_n < 1$. Then instead of (17) we obtain:

$$\overline{e^{i\omega\sigma}} \exp\left[-\frac{\omega^2}{2}\Phi(x)\right], \tag{19}$$

where

$$\Phi(x) = \sum_n \frac{b_n^2}{2q_n^2 c^4} F_n^2(x, \theta_n \alpha_n). \tag{20}$$

Thus, the mean strength of the sound signal (10) drops sharply with increasing frequency ω (according to a Gaussian curve).

B. Propagation of light in a medium with a turbulent metric.

The wave equation in this case is[2]

$$g^{\mu\nu}\partial^2\varphi/\partial x_\mu \partial x_\nu - \Gamma^\mu \partial\varphi/\partial x_\mu = 0, \tag{21}$$

$$\Gamma^\mu = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\, g^{\mu\nu}\right), \tag{22}$$

where g, as usual, is $\det(g_{\mu\nu})$.

We shall regard $g^{\mu\nu}$ as random functions of the variables $x_1, x_2, x_3, x_4 = t$. The physical reasons for such a possibility are considered in (3).

Let us turn to the case when

$$g^{\mu\nu} = g_0^{\mu\nu} + \varepsilon^{\mu\nu}, \qquad \overline{\varepsilon^{\mu\nu}} = 0. \tag{23}$$

From (21), (22), (7), and (8) we obtain the equation for σ:

$$\varepsilon^{\mu\nu}\frac{\partial \overline{S}}{\partial x_\mu}\frac{\partial \overline{S}}{\partial x_\nu} + g_0^{\mu\nu}\left(\frac{\partial \overline{S}}{\partial x_\mu}\frac{\partial \sigma}{\partial x_\nu} + \frac{\partial \overline{S}}{\partial x_\nu}\frac{\partial \sigma}{\partial x_\mu}\right) = 0. \tag{24}$$

In particular, for a plane wave $\overline{S} = t - x$ (we take the speed of light c to be equal to 1), equation (24) takes the form

$$\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} = \frac{1}{2}\varepsilon(x, t), \qquad \varepsilon = \varepsilon^{44} + \varepsilon^{11} - 2\varepsilon^{14}. \tag{25}$$

It has the solution

$$\sigma(x,t) = \frac{1}{4}\int_{t_0}^{t} \varepsilon(t'-\xi, t')\,dt' + \frac{1}{4}\int_{x_0}^{t} \varepsilon(x', \xi + x')\,dx', \tag{26}$$

where $\xi = t - x$. An analogous solution is obtained for the wave $\overline{S} = t + x$. If $\varepsilon(x,t)$ can be represented in the form

$$\varepsilon(x,t) = \sum_{n,m} \varepsilon_{nm} \sin(q_{nx} + \alpha_n) \sin(\omega_m t + \beta_m), \tag{27}$$

where $\varepsilon_{nm}, \alpha_n, \beta_m$ are random quantities, then, for a normal distribution law of ε_{nm}, $\overline{\varepsilon_{nm}} = 0$, $\overline{\varepsilon_{nm}^2} = \frac{1}{2}b_{nm}^2$, and a uniform distribution of α_n, β_m, we obtain

$$\sigma(x,t) = \sum_{n,m} \varepsilon_{nm} F_{nm}(x,t), \tag{28}$$

where the explicit form of $F_{nm}(x,t)$ is not difficult to obtain from (26) and (27). Applying the same reasoning as in point A, we find that the light signal has the form (19), with $\Phi(x) > 0$ now equal to

$$\Phi(x) = \frac{1}{2}\sum_{n,m} b_{nm}^2 F_{nm}^2(x, t, \theta_n\alpha_n, \theta_m\beta_m). \tag{29}$$

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Note: Figure translations are in progress. See original paper for figures.