APPLICATION OF TOPOLOGICAL METHODS TO THE STUDY OF THE CRITICAL REGIME OF A REACTOR

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APPLICATION OF TOPOLOGICAL METHODS TO THE STUDY OF THE CRITICAL REGIME OF A REACTOR

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In the present work, by the topological methods of the monograph of M. A. Krasnosel' skii ([1]), a special system of equations of the form

$$\begin{aligned} x &= \mu A(x, y), \\ y &= B(x, y). \end{aligned} \tag{1}$$

is investigated.

The results obtained are applied to the study of the critical regime of a reactor.

1. We introduce some notation which we shall use. Let E_1 and E_2 be two real Banach spaces with cones K_1 and K_2 ([2]). Denote by $E = E_1 \times E_2$ the Banach space of elements $z = (x, y)$ $(x \in E_1,\ y \in E_2)$ with norm $\|z\| = \|x\| + \|y\|$. Obviously, $K = K_1 \times K_2$ is a cone in the space E.

Let A and B be operators acting respectively from the space E into E_1 and E_2. Then the operator $C = (A, B)$ will act in the space E. Denote by $C(\mu)$ the operator $(\mu A, B)$. It is easy to see that system (1) can be written in the form of a single equation

$$z = C(\mu)z. \tag{2}$$

By P and Q we denote the Fréchet derivatives at zero θ (if they exist) of the operators A and B. By D and $D(\mu)$ we denote respectively the operators (P, Q) and $(\mu P, Q)$. Obviously, the operator $D(\mu)$ is the Fréchet derivative at zero θ of the operator $C(\mu)$. By Q_1 and Q_2 we denote the operator Q respectively on the subspaces E_1 and E_2.

2. In this item we shall formulate two theorems on the computation of rotations of vector fields, analogous to the theorems of Leray and Schauder ([1]).

A solution $z = (x, y)$ of equation (2) will be called a **semiproper vector** of the operator C, if $x \neq \theta$. The corresponding number μ ($\lambda = 1/\mu$) will be called a **semicharacteristic** (**semiproper**) **number** of the operator C. The totality of all semiproper numbers of the operator C will be called the **semispectrum of the operator** C. The multiplicity of the characteristic number μ of the operator

$$P_0 x = P[x, (I - Q_2)^{-1} Q_1 x]$$

will be called the **π-multiplicity** of the semicharacteristic number μ of the linear operator D.

Let $(T_1)_{r_1} \subset E_1$ be the ball $\|x\| \leq r_1$, and $(T_2)_{r_2} \subset E_2$ the ball $\|y\| \leq r_2$. Denote by $(S_1)_{r_1}$ and $(S_2)_{r_2}$, respectively, the spheres $\|x\| = r_1$ and $\|y\| = r_2$.

Theorem 1. *Let the linear operator $D = (P, Q)$ be completely continuous. Let unity not be a point of the spectrum of the operator Q_2. Then the rotation γ of the vector field $z - D(\mu_0)z$ ($\mu_0 \neq 0$) ([1]) on the boundary Γ of the set $(T_1)_{r_1} \times (T_2)_{r_2}$ is equal to the product of the rotations γ_1 and γ_2 of the vector fields $x - \mu_0 P[x, (I - Q_2)^{-1} Q_1 x]$ and $y - Q_2 y$, respectively, on the spheres*

$(S_1)_{r_1}$ and $(S_2)_{r_2}$, i.e. $\gamma = (-1)^\beta \gamma_2$, where β is the sum of the π-multiplicities of all semi-characteristic numbers $\mu \in (0, \mu_0)$ of the operator D.

Theorem 2. Let zero θ be a fixed point of the completely continuous vector field $\Phi = I - C(\mu_0)$ ($\mu_0 \neq 0$) ([1]). Let D be the Fréchet derivative of the operator C at zero θ. Finally, suppose that unity is not an eigenvalue of either the operator $D(\mu_0)$ or Q_2.

Then θ is an ([1]) isolated fixed point of the vector field Φ, and the index ([1]) of this point is equal to $(-1)^\beta \gamma_2$, where β is the sum of the π-multiplicities of all semi-characteristic numbers of the operator D from the interval $(0, \mu_0)$, and γ_2 is the index of zero θ of the field $I - Q_2$ in E_2.

3. We now present a theorem on bifurcation points of operators. We shall call the number μ_0 a point of π-bifurcation of the operator C if, for every $\varepsilon > 0$, there exists a semi-characteristic number μ of the operator C such that $|\mu - \mu_0| < \varepsilon$ and to the number μ there corresponds at least one semi-eigenvector $z : C(\mu)z = z$, with norm $\|z\| < \varepsilon$.

Theorem 3. Let the completely continuous operator C ($C\theta = \theta$) have at zero θ the Fréchet derivative D. Suppose that unity is not a point of the spectrum of the operator Q_2.

Then:

1) Only semi-characteristic numbers of the operator D and the number zero can be points of π-bifurcation of the operator C.

2) Every semi-characteristic number μ_0 of odd π-multiplicity of the operator D is a point of π-bifurcation of the operator C; moreover, at this point there corresponds a continuous branch ([1]) of semi-eigenvectors of the operator C. If it is additionally known that zero θ is an isolated fixed point of the vector field $I - C(\mu_0)$, then the semispectrum of the operator C includes some interval.

4. In the present section we formulate a theorem that is an analogue of the topological principle of M. A. Krasnosel'skii (see ([1]), pp. 244–249, Theorem 1.1).

Theorem 4. Let the positive operator $C = (A, B)$ $(CK \subset K)$ be completely continuous. Suppose that

$$\inf_{x\in\Gamma_1\cap K_1,\, y\in K_2\cap(T_2)_{r_2}} \|A(x,y)\| > 0,$$

where Γ_1 is the boundary of some open set $G_1 \subset E_1$ containing zero θ. Finally, suppose that for any $x \in \Gamma_1 \cap K_1$ the operator $B_{xy} = B(x,y)$ maps the convex set $K_2 \cap (T_2)_{r_2}$ into itself.

Then the operator C has a semi-eigenvector $z = (x, y)$ in the cone K, with $x \in \Gamma_1 \cap K_1,\ y \in K_2 \cap (T_2)_{r_2}$.

5. We now turn to the application of the results obtained. Let the distribution of the neutron flux Φ and of the temperature T over the volume V of the reactor be described by the ([3]) equations

$$\begin{aligned} &\nabla^2\Phi + \mu a(T)\Phi = 0, \qquad \Phi|_\Sigma = 0; \\ &\nabla^2 T + b(T)\Phi = 0, \qquad T|_\Sigma = T_0, \end{aligned} \tag{3}$$

where Σ is the boundary of the reactor, assumed to be sufficiently smooth and convex. In view of the physical meaning ([3]), the coefficients $a(T)$ and $b(T)$ are positive and continuous. The system (3) of differential equations can be replaced by the equivalent system of integral equations

$$\Phi(P) = \mu \int_V K(P,Q)a[T(Q) + T_0]\Phi(Q)\,dQ = \mu A(\Phi, T),$$

$$T(P) = \int_V K(P,Q)b[T(Q) + T_0]\Phi(Q)\,dQ = B(\Phi, T). \tag{4}$$

The system (4) was studied by O. B. Moskalev in paper (3). We investigate it under other restrictions.

Below, the roles of E_1, E_2 and K_1, K_2 are played respectively by the space $C(V)$ of functions continuous on V and the cone $K(V)$ of nonnegative functions of the space $C(V)$.

In paper (3) it is shown that the operator $C = (A, B)$ is positive and completely continuous.

Theorem 5. The points of π-bifurcation of the system (4) can only be the characteristic numbers of the operator

$$A_0\Phi = a(T_0)\int_V K(P,Q)\Phi(Q)\,dQ.$$

Each characteristic number of odd multiplicity of the operator A_0 is a point of π-bifurcation of the system (4). In particular, the characteristic number μ_0 of the operator A_0, to which there corresponds the eigenvector $\Phi_0 \in K_1$, is a point of π-bifurcation of the system (4).

If, in addition, it is known that the kernel $K(P,Q)$ of the system (4) is symmetric and that in some neighborhood of the point $T = 0$ the functions $a(T+T_0)$ and $b(T+T_0)$ have continuous derivatives respectively up to orders n and m inclusive, where

$$a'(T_0) = \ldots = a^{(n-1)}(T_0) = 0, \quad a^{(n)}(T_0) \neq 0$$

and

$$b'(T_0) = \ldots = b^{(m-1)}(T_0) = 0, \quad b^{(m)}(T_0) \neq 0,$$

then for any characteristic number μ_* of odd multiplicity of the operator A_0:

a) if n is an even number and $a^{(n)}(T_0) < 0$, then the half-spectrum of the system (4) completely fills some interval $(1/\mu_1, 1/\mu_*)$, where $\mu_1 < \mu_*$ if $\mu_* < 0$, and $\mu_2 < \mu_*$ if $\mu_* > 0$.

b) if n is an even number and $a^{(n)}(T_0) > 0$, then the half-spectrum of the system (4) completely fills some interval $(1/\mu_2, 1/\mu_*)$, where $\mu_2 > \mu_*$ if $\mu_* < 0$, and $\mu_2 < \mu_*$ if $\mu_* > 0$.

Definition. We shall say that the semi-eigenvectors z of the operator C form a conditionally continuous branch of infinite length if, for the boundary Γ_1 of any bounded open set $G_1 \subset E_1$ containing zero θ, there exists such a semi-eigenvector $z = (x, y)$ of the operator C that $x \in \Gamma_1$.

Theorem 6. Suppose that for all $T \geq 0$

$$b(T+T_0) \leq \alpha + \beta T^q,$$

where $\alpha, \beta > 0,\ q \in (0,1)$.

Then the semi-eigenvectors $z = (\Phi, T) \in K$ of the system (4) form a conditionally continuous branch of infinite length.

If, in addition, it is known that:

a) there exists the limit

$$\lim_{T\to\infty} a(T+T_0) = a_0 < \infty;$$

b) for all $T \geq 0$

$$b(T+T_0) \geq b_0 > 0,$$

then the half-spectrum of the system (4) includes the interval

$$\left(\frac{1}{\mu_0},\ \frac{1}{\mu_0}\frac{a_0}{a(T_0)}\right).$$

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Note: Figure translations are in progress. See original paper for figures.