# SOME NEW ESTIMATES IN THE THEORY OF FINITE AUTOMATA

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## Abstract

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# SOME NEW ESTIMATES IN THE THEORY OF FINITE AUTOMATA

*(Presented by Academician V. M. Glushkov, 5 VI 1965)*

Let a finite system of directed graphs with a common set $S$ of vertices be given,

$$G = \{(S, \Gamma_1), (S, \Gamma_2), \dots, (S, \Gamma_k)\}.$$

Select some $S_0 \subseteq S$ and call it the initial set. We extend in the natural way the mappings $\Gamma_i$ to the set $\Sigma$ of all subsets of the set $S$:

$$\Gamma_i M = \bigcup_{\sigma \in M} \Gamma_i \sigma \qquad (M \subseteq S).$$

By $\widetilde{\Sigma}$ we shall denote the smallest subset of the set $\Sigma$, closed with respect to all mappings $\Gamma_i$, such that $S_0 \in \widetilde{\Sigma}$. Generalizing the definition given in (1), we shall call the transformation of the system $G$ into the system

$$\widetilde{G} = \{(\widetilde{\Sigma}, \Gamma_1), \dots, (\widetilde{\Sigma}, \Gamma_k)\}$$

an optimal determinization of the system $G$.

Everywhere below we shall consider systems of graphs possessing the following orthogonality property:

$$\Gamma_i S \cap \Gamma_j S = \emptyset \qquad (i \neq j).$$

Many problems of automaton synthesis reduce to determinization of an orthogonal system of graphs. In this connection there naturally arises the question of estimating the number $d(G)$ of elements of the set $\widetilde{\Sigma}$.

Denoting by $|A|$ the cardinality of the set $A$, put $n = |S|$, $n_i = |\Gamma_i S|$. By orthogonality,

$$\sum_{i=1}^{k} n_i \le n. \tag{1}$$

Since $\widetilde{\Sigma}$ contains only $S_0$ and subsets of the sets $\Gamma_{iS}$,

$$d(G) \le 2^{n_1} + 2^{n_2} + \dots + 2^{n_k} + 1. \tag{2}$$

Let us now denote by $d_k(n)$ the maximum of $d(G)$ over all orthogonal systems $G$ with fixed $k$ and $n$. Inequality (2) gives for $d_k(n)$ the trivial estimate

$$d_k(n) \le 2^n + 1. \tag{3}$$

In the work (1), Yu. I. Lyubich found that for $k = 1$ the estimate (3) can be replaced by the substantially more exact estimate

$$d_1(n) \le M(n) \qquad (n \ge 6), \tag{4}$$

where

$$M(n) = \max\left\{ m(N_1, N_2, \dots, N_s) + \sum_{k=2}^{s} m(N_1, N_2, \dots, N_k) \right\} + n^2 - 2n + 6,$$

the maximum is taken over all systems $(N_1, N_2, \dots, N_s)$ $(s = 2, 3, \dots)$ of natural numbers satisfying the condition

$$N_1 + N_2 + \dots + N_s \le n;$$

$m( \, , \, )$ denotes the least common multiple. There was also given there a found –

the asymptotic formula obtained by I. V. Ostrovskii

$$\ln M(n) \sim \sqrt{n \ln n} \qquad (n \to \infty) \tag{5}$$

and it was proved that

$$\ln d_1(n) \sim \sqrt{n \ln n} \qquad (n \to \infty).$$

Relying on these results, in the present note we shall establish that estimate (3) can be substantially lowered also for $k > 1$. We shall then use this to obtain asymptotically exact estimates in some problems of automata synthesis. Everywhere in what follows it is assumed, for definiteness, that

$$n_1 \leq n_2 \leq \ldots \leq n_k. \tag{6}$$

**Theorem 1.** *The inequality holds*

$$d(G) \leq (2^{n_1} + 2^{n_2} + \ldots + 2^{n_{k-1}} + 1)(d_1(n_k) + 1). \tag{7}$$

**Proof.** Denote by $\Sigma_1$ the class of all sets of the form $\Gamma_k p S_0$ $(p = 0, 1, 2, \ldots)$. Denote by $\Sigma_2$ the class of all sets of the form $\Gamma_k p M$ $(p = 0, 1, 2, \ldots;\ M \subseteq \Gamma_{iS};\ i = 1, 2, \ldots, k-1)$. Clearly, $\Sigma \subseteq \Sigma_1 \cup \Sigma_2$. But

$$|\Sigma_1| \leq d_1(n_k) + 1, \qquad |\Sigma_2| \leq (2^{n_1} + 2^{n_2} + \ldots + 2^{n_{k-1}})(d_1(n_k) + 1).$$

The theorem is proved.

It is easy to verify that, under conditions (1) and (6), the inequality

$$2^{n_1} + 2^{n_2} + \ldots + 2^{n_{k-1}} \leq 2^{n/2} \qquad (n \geq 8) \tag{8}$$

is valid.

From (4), (7), and (8) it follows

**Corollary 1.** *The inequality holds*

$$d(G) \leq (2^{n/2} + 1)(M(n) + 1) \qquad (n \geq 8). \tag{9}$$

Since this estimate uses no information about the orthogonal system $G$ except the number $n$, the inequality

$$d_k(n) \leq (2^{n/2} + 1)(M(n) + 1) \qquad (n \geq 8) \tag{10}$$

holds.

We note that knowledge of the structure of the system $G$ makes it possible in some cases, using the results obtained in ([1]), to improve estimate (9). For example, the following holds:

**Corollary 2.** *If the graph* $(\Gamma_{kS}, \Gamma_k)$ *is strongly connected,* then

$$d(G) \leq (2^{n/2} + 1)(n^2 - 2n + 4) \qquad (n \geq 8).$$

Let us apply the results obtained to derive estimates in two problems of automata synthesis. Everywhere in what follows Moore automata are meant.

Let $E$ be a regular event (see ([3],[4])). Denote by $f(E)$ the number of states of the minimal automaton representing the event $E$. Next put $\varphi(n) = \max f(E)$, where the maximum is taken over all events $E$ enumerable** by automata with $n$ states. We shall be interested in the question of the rate of growth of $\varphi(n)$. This question was posed by V. A. Uspenskii (see, for example, ([6])) and for Mealy automata was completely solved independently by Yu. L. Ershov ([7]) and O. B. Lupanov ([5]).

* For the definition of a strongly connected graph, see, for example, ([2]).

** For the definition of an event enumerable by an automaton, see, for example, ([5]).

**Theorem 2.** The following asymptotic equality holds:

$$\log_2 \varphi(n) \sim n/2 \qquad (n \to \infty). \tag{11}$$

**Proof.** There is a well-known algorithm (see, for example, (7)) for constructing, from a given automaton $\mathfrak{A}$, an automaton $\widehat{\mathfrak{A}}$ representing the event that is enumerated by the automaton $\mathfrak{A}$. In our terms this algorithm is the determinization of the system of graphs $G$ obtained if we denote by $k$ the number of letters of the output alphabet of the automaton $\mathfrak{A}$, by $S$ the set of states of the automaton $\mathfrak{A}$, and by $\Gamma_i$ the mapping specified by those transitions in the automaton $\mathfrak{A}$ for which the $i$-th alphabet symbol is supplied at the output. In the case of a Moore automaton this system is, obviously, orthogonal. Therefore

$$\varphi(n) \le \max_k d_k(n).$$

Taking (5) and (10) into account, we obtain

$$\log_2 \varphi(n) \le \frac{n}{2}[1 + o(1)].$$

The validity of (11) now follows from the following result of G. M. Kornelevich (6):

$$\varphi(n) \ge 2^{[n/2]} \qquad (n \ge 4).$$

Theorem 1 also makes it possible to obtain the asymptotics of the maximum amount of memory specified by a regular expression of length $n$ ($n \to \infty$) over a $k$-letter alphabet for arbitrary $k > 1$. For $k = 1$ the asymptotics was obtained in (8). By the length of a regular expression $R$ is meant the number of occurrences in it of letters of the alphabet. The weight $V(R)$ of a regular expression is the number of states of the minimal automaton representing the corresponding event. Put

$$\psi_k(n) = \max V(R),$$

where the maximum is taken over all regular expressions of length $n$ over a $k$-letter alphabet.

V. M. Glushkov established (8) that

$$\psi_k(n) \leq 2^n + 1.$$

**Theorem 3.** For $k \geq 2$ the following asymptotic equality holds:

$$\log_2 \psi_k(n) \sim n/2 \qquad (n \to \infty).$$

**Proof.** The known algorithm (4) for constructing an automaton representing an event consists in the following: from a given regular expression of length $n$, a certain orthogonal system of graphs $G$ is constructed on a set $S$ of cardinality $n+1$, and then it is determinized. Consequently,

$$\log_2 \psi_k(n) \leq \frac{n}{2}[1 + o(1)].$$

To complete the proof it is enough to construct a regular expression over an alphabet of two symbols $x$ and $y$, whose length is an arbitrary even $n = 2p$ and whose weight is no less than $2^{p-1}$. Such an expression is

$$R_p = (\{x\}(\bar{x}\{y\})^{-p-1}x).$$

We note that the regular expression $R_p$ is closed (9). In (9) it was found that for $k = 1$ closed regular expressions of length $n$ have weight

$$\leq n^2 - 2n + 4 \qquad (n \geq 5),$$

i.e., substantially smaller than $\psi_1(n)$, since

$$\ln \psi_1(n) \sim \sqrt{n \ln n} \qquad (n \to \infty).$$

At the same time, we have seen that for $k > 1$ the maximum of the weight is attained asymptotically on closed regular expressions.

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*Note: Figure translations are in progress. See original paper for figures.*