POSITIVE HARMONIC FUNCTIONS ON NILPOTENT GROUPS

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Abstract

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POSITIVE HARMONIC FUNCTIONS ON NILPOTENT GROUPS

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Let $p(g)$ be a nonnegative function on a discrete group G. Put

$$Lf(x) = \sum_{g \in G} p(g) f(xg) \quad (x \in G).$$

A function $u(x)$ is called **harmonic** if $Lu(x) = u(x)$, and superharmonic if $Lu(x) \leqslant u(x)$.

In the case when the group G is abelian, the cone of positive harmonic functions was described in (^{1}). In (^{2}) it was shown that on nilpotent groups there exist no bounded harmonic functions distinct from constants (under certain natural restrictions on the function $p(g)$). The main result of the present note is the following:

Theorem. Let G be a nilpotent group. Suppose that for every $g \in G$ there can be found a finite number of elements $g_1, g_2, \dots, g_n$ such that $g = g_1 g_2 \dots g_n$ and $p(g_i) > 0$ (condition S). Then every positive harmonic function on G is constant on the cosets with respect to the commutator subgroup R of the group G.

By virtue of this theorem, the study of harmonic functions on nilpotent groups reduces to their study on abelian groups.

Put $f_n \to f$ if $f_n(x) \to f(x)$ for all x. If $f_n > 0$, then

$$Lf(x) \leqslant \lim_{n \to \infty} Lf_n(x). \tag{1}$$

Hence it is clear that the limit of superharmonic functions is a superharmonic function (the limit of harmonic functions is not necessarily a harmonic function). Consider now the set A of all positive harmonic functions equal to one at the identity e of the group G, and close it. Denote the resulting set by A'. Obviously, A' is convex. From condition S it follows that

$$0 < b < a(x)/a(xg) \quad (a(x) \in A'). \tag{2}$$

It follows from this that A' is bicompact.

A point x of a subset D of a linear space is called **extreme** if from the fact that $x = px_1 + qx_2$ $(x_1, x_2 \in D)$, $p > 0$, $q > 0$, $p + q = 1$, it follows that $x_1 = x_2 = x$.

The following holds.

Choquet' s theorem (^{3}). Let F be a bicompact metrizable subset of a locally convex linear topological space, and let E be the set of its extreme points. If $a \in F$, then

$$a = \int_E k\mu(dk),$$

where μ is a normalized measure defined on the Borel subsets of the set E.

We apply Choquet' s theorem to the set A'. Let us show that, if $f(x)$ is a harmonic function, then the corresponding measure μ is concentrated on harmonic functions. Indeed, from inequality (1) it is easy to derive that

$$Lf(x) \leq \int_E Lk(x)\,\mu(dk). \tag{3}$$

Put $l_x(k) = Lk(x) - k(x)$. From (1) it is clear that

$$Lf(x) - f(x) \leq \int_E l_x(k)\,\mu(dk).$$

Since $Lf(x) = f(x)$, it follows that $l_x(k) \leq 0$ on a set of measure zero. But the set of elements x is countable; consequently, $Lk(x) = k(x)$ for all x on a set of measure 1. Thus the problem of finding all positive harmonic functions reduces to the problem of finding the extreme points of the set A' which are harmonic functions. In what follows, by $k(x)$ we shall mean an arbitrary element of this type. If $l(x)$ is a harmonic function, then $l(gx)$ is also a harmonic function.

Lemma 1. Denote by Z the center of the group G. If $z \in Z$, then $k(xz) = k(x) \cdot k(z)$.

Proof. From (2) it is seen that $bk(xz) < k(x)$ (b does not depend on x). Therefore

$$k(x) = p\,\frac{k(xz)}{k(z)} + q\,\frac{c(xz)}{c(z)} \qquad (p = bk(z),\ q = c(z)).$$

Observe that $p > 0,\ q > 0,\ p + q = 1$; $k(xz)/k(z)$ and $c(xz)/c(z)$ belong to A'. Since $k(x)$ is an extreme point, $k(xz)/k(z) = k(x)$.

Lemma 2. If $z \in P = R \cap Z$, then $k(z) = 1$.

Proof. Put $p_x = lxl^{-1}x^{-1}$. Let $p_u \in P,\ p_v \in P$. We have

$$p_{uv} = luvl^{-1}v^{-1}u^{-1} = lul^{-1}p_{vu}^{-1} = lul^{-1}u^{-1}p_v = p_{up}v.$$

Hence, if $p_x \in P$, then $p_{x^n} = p_x^n$.

Consider the closed cone K generated by the functions $k(l^n x)$. Clearly, every function $h \in K$ satisfies the condition $h(xz) = k(z)h(x)$. The cone K is invariant under left shift by l. Select from K the set H of functions h for which $h(t) = 1$. Consider the transformation

$$T_{lh}(x) = h(lx)/h(l).$$

Since T_l is continuous, by the Schauder-Tikhonov fixed-point theorem ((4), p. 493), there exists such an h_1 that $T_{lh}1 = h_1$, or

$$h_1(x) = h_1(lx)/h_1(l).$$

By virtue of (2), for every n,

$$0 < b < h_1(x^n l)/h_1(x^n). \tag{4}$$

On the other hand, $lx^n = p_{x^n}x^{nl}$; consequently,

$$h_1(lx^n) = k(p_x^n) \times h_1(x^n l),$$

and

$$\frac{h_1(x^n l)}{h_1(lx^n)} = \frac{1}{k(p_x)^n} h_1(l). \tag{5}$$

From (4) and (5) it follows easily that $k(p_x) = 1$. This proves the lemma.

Let now G be an arbitrary group and N some normal divisor of it. Put

$$p_1(g_1) = \sum_{g \in G} p(g)$$

(S is the adjacent class corresponding to g_1). If the harmonic function $f(x)$ ($x \in G$) is constant-on the adjacency classes of G with respect to N, then on G/N the function $f(x)$ induces a function $f_1(y)$ $(y \in G/N)$, harmonic with respect to $p_1(g_1)$.

Let $G_1 = G/P$. On G_1 the function $k(x)$ induces a function $k_1(y)$ $(y \in G_1)$. The function $k_1(y)$ is constant on the adjacency classes of G_1 with respect to $P_1 = Z_1 \cap R/P_1$, where Z_1 is the center of the group G_1. Therefore the function $k(x)$ is constant on the adjacency classes of G with respect to the full inverse image of the group P_1 under the mapping $G \to G_1$. Let $P_0 = P$, $R_0 = R$, $G_0 = G$, $R_{i+1} = R_i/P_i$, $G_{i+1} = G_i/P_i$, Z_i be the center of the group G_i, and $P_i = R_i \cap Z_i$. From the nilpotency of the group G it follows that for every x there is an i such that the image of x under the natural mapping $G \to G_i$ will be the identity of the group G_i. Hence it follows directly that every harmonic function on G is constant on the adjacency classes of G with respect to R.

In the case where the group G is abelian, it was shown in [1] that the extreme elements of the set A' satisfy the relation

$$k(x_1 x_2) = k(x_1) k(x_2). \tag{6}$$

From what has been proved it follows that relation (6) also holds for nilpotent groups.

Now let G be a locally compact topological group satisfying the second countability axiom, let μ be a left-invariant measure on it, and let $p(g)$ be a nonnegative measurable function. Put

$$Lf(x) = \int_G p(g) f(xg)\, d\mu,$$

where $f(x)$ is a locally integrable function.

Let

$$p_0(g) = p(g), \qquad p_{i+1}(g) = \int_G p(g_1) p_i(g_1^{-1} g)\, \mu(dg_1).$$

We shall say that $f_n \to f$ if

$$\int_V f_n\, d\mu \to \int_V f\, d\mu,$$

where V is an arbitrary open compact set. The theorem proved in the note remains valid if condition S is replaced by the following:

$$\int_U \sum_{i=0}^{\infty} p_i(g)\, \mu(dg) > 0$$

for an arbitrary compact open set U. We first prove that the set A of positive harmonic functions $a(x)$ such that

$$\int_{U_1} a(x)\,\mu(dx) = 1,$$

where U_1 is some fixed neighborhood of the identity, will be compact. Compactness will follow from the fact that

$$\int_U a(x)\,\mu(dx) < c \int_{U_1} a(x)\,\mu(dx) \quad (a \in A),$$

where c is a certain constant not depending on a.

We prove this inequality. Suppose that $U \cdot U^{-1} \subset U_1$. It is easy to show that

$$c_1 \mu(V \cdot g) < \mu(V)$$

(V is an open subset of the compact set W, $g \in W$, and c_1 is a constant depending only on W) ([5], p. 49). Hence it follows that

$$\int_{U_1} a(x)\,\mu(dx) = \int_{U_1}\int_G a(xg)p_i(g)\,\mu(dx)\mu(dg) \geq \int_{U_1}\int_U a(xg)p_i(g)\,\mu(dx)\mu(dg) =$$

$$= \int_U p_i(g)\left(\int_{U_1} a(xg)\,\mu(dx)\right)\mu(dg) > c_1 \int_U p_i(g)\,\mu(dg) \cdot \int_U a(y)\,\mu(dy).$$

But for some i the integral $\int_{\breve{U}} p_i(g)\,\mu(dg) > c_2 > 0$, whence it is clear that

$$\int_{\breve{U}_1} a(x)\,\mu(dx) > \frac{1}{c_1 c_2}\int_{\breve{U}} a(y)\,\mu(dy).$$

After the proof of compactness, the proof of the theorem carries over without essential changes.

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REFERENCES CITED

[1] J. L. Doob, J. L. Snell, R. E. Williamson, *Contributions to Probability and Statistics*, 1960, p. 182.
[2] E. B. Dynkin, M. B. Malyutov, *DAN*, 137, No. 5 (1961).
[3] G. Choquet, P.-A. Meyer, *Ann. Inst. Fourier*, 13, 139 (1963).
[4] N. Dunford, J. T. Schwartz, *Linear Operators*, Moscow, 1962.
[5] A. Weil, *Integration in Topological Groups and Its Applications*, 1950.

Note: Figure translations are in progress. See original paper for figures.