OPTIMALITY CONDITIONS FOR SYSTEMS CONTAINING ELEMENTS WITH DISTRIBUTED PARAMETERS

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Abstract

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OPTIMALITY CONDITIONS FOR SYSTEMS CONTAINING ELEMENTS WITH DISTRIBUTED PARAMETERS

(Presented by Academician L. S. Pontryagin on 4 III 1966)

L. S. Pontryagin' s maximum principle is an effective method for investigating optimal processes described by ordinary differential equations or by differential-difference equations ([1]). However, optimal systems containing elements with distributed parameters, in which the processes are described by partial differential equations, are of practical interest.

Let the process in the controlled object be described by the boundary-value problem

$$u_{it} - a_i^2 u_{ixx} = f_i(t, x, u_1, \dots, u_n, u_{1x}, \dots, u_{nx}, \alpha),$$
$$0 < t \le T, \qquad 0 < x < 1; \tag{1}$$

$$u_{ix}^0 = \varphi_i^0(t, u_1^0, \dots, u_n^0, y_1^0, \dots, y_l^0, \beta^0),$$
$$u_{ix}^1 = \varphi_i^1(t, u_1^1, \dots, u_n^1, y_1^1, \dots, y_m^1, \beta^1); \tag{2}$$

$$u_i(0, x) = u_{i0}(x), \qquad i = 1, \dots, n, \tag{3}$$

where a_i are real constants; $u_i^k = u_i(t, k)$, $u_{ix}^k = u_{ix}(t, k)$. The functions f_i and φ_i^k are continuous in t, continuously differentiable in x, and have continuous second-order derivatives with respect to the remaining arguments jointly; $u_{i0}(x)$ are continuous on $[0, 1]$. The control parameters α and β^k take values from certain open or closed domains A and B^k of r- and r^k-dimensional Euclidean spaces. The functions $\alpha(t, x)$ satisfy the differential equations

$$\dot{y}_j^0 = \psi_j^0(t, u_1^0, \dots, u_n^0, y_1^0, \dots, y_l, \beta^0), \qquad j = 1, \dots, l,$$
$$\dot{y}_k^1 = \psi_k^1(t, u_1^1, \dots, u_n^1, y_1^1, \dots, y_m^1, \beta^1), \qquad k = 1, \dots, m, \tag{4}$$

with initial conditions

$$y_j^0(0) = y_{j0}^0, \qquad y_k^1(0) = y_{k1}^1, \quad j = 1, \dots, l; \quad k = 1, \dots, m, \tag{5}$$

where the functions ψ_j^k satisfy the same conditions as φ_j^k.

Admissible controls are functions $w = \{\alpha(t,x), \beta^0(t), \beta^1(t)\}$ satisfying the following requirements: 1) $\alpha(t,x)$ and $\beta^k(t)$ take values respectively in A and B^k; 2) $\beta^k(t)$ are piecewise continuous, with no more than a finite number of discontinuity points; 3) the functions $\alpha(t,x)$ have no more than a finite number of lines of discontinuity in the domain D $(0 \le x \le 1,\ 0 \le t \le T)$. At the same time, for individual components of the vector w, the conditions of the problem may prescribe in advance the form of their dependence on x and t. For example, the components of the vectors $\alpha(t,x)$ must consist of two groups $\alpha_i,\ i = 1, \dots, q$, and $\alpha_j,\ j = q+1, \dots, r$, where the α_i may depend only on x, and the α_j only on t.

It is assumed that all the given functions in the boundary-value problem (1)–(5), in addition to the properties listed above, also satisfy conditions under which each admissible control corresponds to a unique solution (classical or generalized) of this problem.

We shall say that an admissible control transfers the system from the state $\{(2),(5)\}$ to the state (6) if the solution of problem (1)–(5) corresponding to it satisfies the conditions

$$\Phi_\alpha(T, x, u(T,x)) = 0, \qquad \alpha = 1, \dots, \nu;$$

$$\int_0^1 \Psi_\beta(T, x, u(T,x))\, dx = c_\beta, \qquad \beta = 1, \dots, \mu,$$

$$y_\gamma^0(T) = y_{\gamma 1}^0, \qquad \gamma = 1, \dots, d \le l; \qquad y_\delta^1(T) = y_{\delta 1}^1, \qquad \delta = 1, \dots, e \le m, \tag{6}$$

where c_β, y_{i1}^k are real constants; Φ_α and Ψ_β are functions continuous in x and having continuous first-order derivatives with respect to u_i and t.

It is required, among all admissible controls transferring the system from the state $\{(2),(5)\}$ to the state (6), to find a control such that the solution of problem (1)–(5) corresponding to it realizes the minimum of the functional

$$S = \sum_{i=1}^n \left\{ \int_0^T \left[A_i^0(t) u_i^0 + A_i^1(t) u_i^1\right] dt + \int_0^1 B_i(x) u_i(T,x)\, dx + \iint_D C_i(t,x) u_i(t,x)\, dx\, dt \right\} + \sum_{j=d+1}^{l} c_j^0 y_j^0(T) + \sum_{k=e+1}^{m}$$

where A_i^k, B_i, and $C_i(t,x)$ are given continuous functions; c_i^k are real constants; the value of T is not fixed in advance.

The special case of this problem, when T is fixed and the final state of the system is free, was considered in [2].

If there exists only a finite number of admissible controls transferring the system from the state $\{(2),(5)\}$ to the state (6), then the formulated problem will not be variational. In order to apply variational methods, one must assume a sufficient completeness of the class of admissible controls. Let this completeness be determined by the following property.

Take arbitrary admissible controls $w(t,x)$ and $w_1(t,x)$, defined in the domains D $(0 \le x \le 1,\ 0 \le t \le T)$ and D_1 $(0 \le x \le 1,\ 0 \le t \le T)$, and transferring the system from the state $\{(2),(5)\}$ to the state (6). Then, for arbitrarily small $\varepsilon > 0$, it is possible to specify an admissible control $w_\varepsilon(t,x)$, defined in the domain D_ε $(0 \le x \le 1,\ 0 \le t \le T_\varepsilon)$, such that:

1) it transfers the system from the state $\{(2),(5)\}$ to the state (6), with conditions (6) being satisfied at $t = T_\varepsilon$;
2) in the domain DD_ε it is defined by the formula

$$w_\varepsilon(t,x) = \begin{cases} w_1(t,x), & \text{for } (t,x) \in G_\varepsilon, \\ w(t,x), & \text{for } (t,x) \in DD_\varepsilon - G_\varepsilon, \end{cases}$$

where G_ε $(G_\varepsilon \subset D)$ is an arbitrary prescribed domain of area ε;

3) the inequality $|T - T_\varepsilon| \le L\varepsilon$ is satisfied, where the constant L does not depend on ε and is determined by the controls w and w_1.

Introduce the notation

$$H(t,x,u,v,a) = \sum v_i f_i, \qquad h^k(t,u^k,v^k,y^k,z^k,\beta^k) = \sum a_i^k v_i^k \varphi_i^k + \sum z_j^k \psi_j^k, \quad k = 0,1,$$

where $v^0 = v(t,0)$, $v^1 = v(t,1)$.

We define the auxiliary functions $v(t,x)$ and $z^k(t)$ by the boundary-value problem

$$v_{it} + a_i^2 v_{ixx} = -\frac{\partial H}{\partial u_i} + \frac{d}{dx}\left(\frac{\partial H}{\partial u_{ix}}\right) + C_i(t,x), \qquad 0 \le t < T, \quad 0 < x < 1,$$

$$v_i(T,x) = -B_i(x) - \sum_{\alpha=1}^{\nu} A_\alpha(x) \frac{\partial \Phi_\alpha(T,x,u(T,x))}{\partial u_i} - \sum_{\beta=1}^{\mu} b_\beta \frac{\partial \Psi_\beta(T,x,u(T,x))}{\partial u_i},$$

$$a_i^2 v_{ix}^0 = \frac{\partial h^0}{\partial u_i^0} + \frac{\partial H}{\partial u_{ix}}\bigg|_{x=0} + A_i^0(t), \qquad a_i^2 v_{ix}^1 = \frac{\partial h^1}{\partial u_i^1} + \frac{\partial H}{\partial u_{ix}}\bigg|_{x=1} - A_i^1(t), \qquad i = 1, \dots, n, \tag{7}$$

$$\dot z_j^0 = -\frac{\partial h^0}{\partial y_j^0}, \qquad \dot z_k^1 = -\frac{\partial h^1}{\partial y_k^1}, \qquad j = 1, \dots, l; \quad k = 1, \dots, m,$$

$$z_j^0(T) = -c_j^0, \qquad z_k^1(T) = c_k^1, \qquad j = d+1, \dots, l; \quad k = l+1, \dots, m.$$

Here $A_i^k(t)$, $B_i(x)$, $C_i(t,x)$, and c_j^k are taken from the functional S, while $A_\alpha(x)$ and b_β are not yet defined.

Let $w(t,x)$ be an arbitrary admissible control defined in the domain D and carrying the system from the state $\{(2),(5)\}$ into the state (6), and let $u(t,x)$, $y^k(t)$ and $v(t,x)$, $z^k(t)$ be the corresponding solutions of the boundary-value problems (1)–(5) and (7). We shall say that $w(t,x)$ satisfies the maximum conditions if, for any admissible control $w(t,x) = \{a(t,x), \beta^0(t), \beta^1(t)\}$, defined in the domain D_1 $(0 \le x \le 1,\ 0 \le t \le T_1)$, the inequalities

$$\iint_G [H(t,x,u(t,x),v(t,x),\alpha_1) - H(t,x,u(t,x),v(t,x),\alpha)]\,dx\,dt \le 0,$$

$$\int_0^\tau [h^k(t,u^k(t),v^k(t),y^k(t),z^k(t),\beta_1^k) - h^k(t,u^k(t),v^k(t),y^k(t),z^k(t),\beta^k)]\,dt \le$$

$$\le 0, \qquad k = 0,1,$$

are satisfied, where $\tau = \min\{T, T_1\}$, and $G = DD_1$.

Theorem. Let $w(t,x)$ be an admissible control, defined in the domain D and carrying the system from the state $\{(2);(5)\}$ into the state (6), and let the functions $u(t,x)$, $y^k(t)$, $v(t,x)$, $z^k(t)$ be the corresponding solutions of the boundary-value problems (1)–(5) and (6) for certain $A_\alpha(x)$, b_β.

Then, for the optimality of this control, it is necessary that:

1) the control $w(t,x)$ satisfy the maximum conditions;

2) all the listed functions and constants satisfy the equality

$$\frac{dS}{dT} + \int_0^1 \left[\sum_{\alpha=1}^{\nu} A_\alpha(x)\frac{d\Phi_\alpha}{dT} + \sum_{\beta=1}^{\mu} b_\beta \frac{d\Psi_\beta}{dT}\right] dt - \sum_{j=1}^{d} z_j^0(T)\dot y_j^0(T) +$$

$$+\sum_{k=1}^{l} z_k^1(T)\dot{y}_k^1(T) = 0.$$

It is easy to verify (2) that the conditions of the theorem determine, generally speaking, isolated controls among which the optimal ones may be found.

An analogous result is obtained if, instead of (2), the conditions

$$u_i^0 = \varphi_i^0(y_1^0, \dots, y_l^0), \qquad u_i^1 = \varphi_i^1(y_1^1, \dots, y_m^1), \qquad i = 1, \dots, n.$$

are taken.

If, instead of S, the optimality criterion is the functional

$$I = \iint_D f_0(t, x, u_1, \dots, u_n, u_{1x}, \dots, u_{nx}, \alpha)\, dx\, dt,$$

then, in order to obtain the optimality conditions, one must introduce an auxiliary variable u_0 in the same way as was done in (2).

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[2] A. I. Egorov, *Automation and Remote Control*, **26**, 7, 1188 (1965).

Note: Figure translations are in progress. See original paper for figures.