THE CLOSED IMAGE OF A METRIC SPACE CAN BE CONDENSED ONTO A METRIC ONE

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THE CLOSED IMAGE OF A METRIC SPACE CAN BE CONDENSED ONTO A METRIC ONE

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The main result of the paper is formulated in the title. After its proof, a number of consequences will be derived from it. It should be noted at once that the proposed assertion is of interest only in the nonseparable case—any continuous image of a space with a countable base can be condensed onto a space with a countable base (the second assertion). This result is simpler; nevertheless, it carries quite serious information—from it, for example, it follows immediately that a bicompactum that is a continuous image of a space with a countable base has a countable base. Hence, in turn, an additional formula for the weight of bicompacta can be extracted (see ([2])).

A few general remarks on the range of problems. When one space can be condensed onto another (more precisely, when each space from a class A can be condensed onto some space from a class B) is one of the typical questions of general theory, whose subject is the study of relations between classes of spaces effected by mappings of various kinds.

The choice of the special question: "When can a space be condensed onto a metric one?"is justified by tendencies in this area: metric spaces occupy a central place in the hierarchy of all topological spaces.

Let us note that the inverse problem is of no interest—every space is a condensation of a discrete metric space. On the contrary, all nonmetrizable bicompacta, as is known, cannot be condensed onto metric ones.

Finally, let us note that the present paper is adjacent to a number of publications on continuous decompositions of spaces into closed sets, not connected with any a priori restrictions on the space of the image. The nontrivial point here is the search, in a decomposition, for a bicompact element. The existence of such an element was proved by the author for the case of complete paracompact spaces, and consequently also for complete metric spaces ([3]). N. Lashnev then obtained the corresponding result for arbitrary metric spaces ([6]). Progress did not stop there ([5]); however, we shall need precisely the last assertion.

Proof of the main assertion. Let $f : X \to Y$ be a closed mapping, X a metric space with metric ρ, and Y a T_1-space (with topology τ_0). Then Y satisfies all separation axioms up to paracompactness. By a theorem of N. Lashnev, there exists a set $N = \bigcup_{i=1}^{\infty} N_i \subseteq Y$ such that: a) all N_i are discrete in Y; b) for all $y \in Y \setminus N$, $f^{-1}y$ is compact. On the set of points of the space Y we now introduce a certain topology τ, generally speaking weaker than τ_0. We indicate a defining system of its neighborhoods.

First of all, all sets open in τ_0 are declared to be neighborhoods of the points belonging to them from $Y \setminus N$. The neighborhoods of points of N are defined by induction. We shall assume that $N_{n_1} \subseteq N_{n_2}$ for $n_2 > n_1$. Put $\gamma_0 = \{(Y)\}$. Suppose that, for each $n = 0, 1, \dots, k$, a system γ_n of sets open in τ_0 has already been defined, so that: 1) distinct elements of γ_n are pairwise disjoint; 2) γ_n covers N_n; 3) if $U \in \gamma_{n_2}$ and $U \cap (N_{n_2} \cap G_{n_1}) \neq \Lambda$, where $k \geq n_2 > n_1$ and $G_{n_1} = \bigcup_{V \in \gamma_{n_1}} V$, then $[U] \subseteq G_{n_1}$.

As γ_{k+1} we take some discrete system of τ_0-open neighborhoods of the points of the set N_{k+1} such that: a) if $y \in N_{k+1}$, $U \in \gamma_i$, where $i \leq k$, $U \ni y$, and $V \ni y$, $V \in \gamma_{k+1}$, then $[V] \subseteq U$; b) $f^{-1}V \subseteq O_{1/(k+1)}(f^{-1}y)$ for $y \in N_{k+1}$, $V \ni y$, $V \in \gamma_{k+1}$. Obviously, the family $\{\gamma_i,\ 0 \leq i \leq k+1\}$ again satisfies conditions 1), 2), 3), and the construction can be continued. The elements of the system

$$\gamma = \bigcup_{i=1}^{\infty} \gamma_i$$

we shall regard as neighborhoods (in τ) of the points belonging to them. Thus neighborhoods are defined for all points of the set Y. Let us verify that their totality, extended in accordance with the condition: if $U \supseteq V$ and V is a neighborhood of the point x, then U is also a neighborhood of x, satisfies the axioms of a defining system of neighborhoods.

A. It is clear that the intersection of any two neighborhoods of an arbitrary point of $Y \setminus N$ is a neighborhood of this point. The same may be said also of neighborhoods of points of N belonging to γ—indeed, any two of them either do not intersect or one is contained in the other.

B. It remains to verify that, for every point $y \in Y$ and every one of its neighborhoods Oy, there is a neighborhood $O_1 y$ such that every point $y' \in O_1 y$ enters Oy together with some neighborhood Oy' of its own.

Two cases are possible.

I. If $y \in N$ and $Oy \in \gamma$, then the assertion is obvious—then Oy is a neighborhood of each of its points and one may, consequently, put $O_1 y = Oy' = Oy$.

II. Let $y \in Y \setminus N$. Then $f^{-1}y$, by assumption, is compact. Therefore

$$\rho(f^{-1}y, X \setminus f^{-1}Oy) > 0.$$

Put $\varepsilon = \rho(f^{-1}y, X \setminus f^{-1}Oy)/2$ and consider

$$O_\varepsilon(f^{-1}y) = \{x \in X \mid \rho(x, f^{-1}y) < \varepsilon\}.$$

By virtue of the closedness of f, there is $O_1y \in \tau_0$, $O_1y \ni y$, such that $f^{-1}O_1y \subseteq O_\varepsilon(f^{-1}y)$. We shall show that O_1y is the required neighborhood of the point y in τ. Let $y' \in O_1y$ be any point. If $y' \in Y \setminus N$, then one may put $Oy' = O_1y$. Let $y' \in N$. Choose an integer $k > 0$ from the conditions $y' \in N_k$ and $1/k < \varepsilon$, and denote by V' the element of the system γ_k containing the point y'. Then

$$f^{-1}V' \subseteq O_{1/k}(f^{-1}y') \subseteq O_\varepsilon(f^{-1}y') \subseteq O_\varepsilon(O_\varepsilon(f^{-1}y)) \subseteq$$

$$\subseteq O_{2\varepsilon}(f^{-1}y) \subseteq f^{-1}Oy.$$

Hence it follows that $V' \subseteq Oy$. Thus, as Oy' one may take V'. Assertion B is verified. The verification of the remaining axioms is trivial.

The topological space that the system of neighborhoods we have constructed defines on the set Y will be denoted by $\widetilde{Y}$. Clearly, $\widetilde{Y}$ satisfies the Hausdorff separation axiom.

We denote the natural one-to-one mapping $Y \to \widetilde{Y}$ by φ. Obviously, φ is continuous—the inverse images under φ of all the neighborhoods we have defined are open in Y.

We adopt a convention convenient for what follows. We shall say that a neighborhood Oy of a point y in $\widetilde{Y}$ has diameter $< \varepsilon$ if $f^{-1}Oy \subseteq O_\varepsilon(f^{-1}y)$. Clearly, every point $y \in \widetilde{Y}$ has neighborhoods of arbitrarily small diameter. We shall show that in $\widetilde{Y}$ there exists a fundamental set of coverings (see (4)).

Denote by F_n the closure in the body τ_0 of the system γ_n: $F_n = [G_n]_Y$, $n = 1, 2, \dots, \infty$; by λ_n denote the union of the system γ_n and the collection of all neighborhoods of diameter $< 1/n$ of points of $Y \setminus G_n$ that do not meet those elements of the system γ_{n+1} which contain points of N_n. Let us note at once that the stars (in τ) of the elements of the system λ_n form an open covering of the space $\tilde{Y}$, because for every point $y \in \tilde{Y}$ there is in λ_n a neighborhood containing it, as follows from the definition of $\{\gamma_n\}$.

We shall show that the system

$$\eta = \{\lambda_n \mid n = 1, 2, \dots, \infty\}$$

satisfies condition

(Φ). Whatever the point $y \in Y$ and its neighborhood O_y in $(\tilde{Y})$ may be, there are a neighborhood O_1y (in τ) and a number n such that $\lambda_n(O_1y) \subseteq O_y$.

There are two possible cases.

1) $y \in N$. Then $y \in N_{k_1}$ for some k_1. We may suppose that $O_y \in \gamma$; then $O_y \in \gamma_{k_2}$ for some k_2. Choose an integer $k > 0$, $k > k_1$, $k > k_2$. Then $y \in N_k$, $y \in N_{k+1}$; in γ_k, γ_{k+1} there is one element in each containing y. Denote them respectively by $V_k(y)$, $V_{k+1}(y)$. From the definition of λ_k it follows easily that $V_{k+1}(y)$ meets only one element of the system V_k—namely $V_k(y)$. Moreover $V_k(y) \subseteq O_y$—this follows from the restrictions imposed on k, y, and from the definition of the sequence $\{\gamma_n\}$. Consequently, $\lambda_{k+1}(V_k) \subseteq V_k(y) \subseteq O_y$, as required.

2) $y \in Y \setminus N$. Then $\rho(f^{-1}y, X \setminus f^{-1}O_y) > 0$, since $f^{-1}y$ is compact. Put

$$\varepsilon = \rho(f^{-1}y, X \setminus f^{-1}O_y)/2.$$

Thus $O_{2\varepsilon}(f^{-1}y) \subseteq f^{-1}O_y$. From the closedness of the mapping $f : X \to Y$ it follows that there is a neighborhood O'_y (in Y) such that

$$f^{-1}(O'_y) \subseteq O_\varepsilon(f^{-1}y).$$

Put

$$\varepsilon' = \rho(f^{-1}y, X \setminus f^{-1}(O'_y)).$$

Then $0 < \varepsilon' \leq \varepsilon$. Choose the number k from the condition $1/k < \varepsilon'/2$. As $O_1 y$ take any neighborhood (in Y and, hence, in $\tilde{Y}$) of the point y for which

$$f^{-1}O_1 y \subseteq O_{\varepsilon'/2}(f^{-1}y).$$

We shall prove that $\lambda_k(O_1 y) \subseteq O_y$.

Indeed, let $V \in \lambda_k$, $V \cap O_1 y \neq \Lambda$, and $y_1 \in V \cap O_1 y$. By the definition of λ_k, for some point $y' \in V$

$$f^{-1}V \subseteq O_{1/k}(f^{-1}y').$$

We have $f^{-1}y_1 \subseteq f^{-1}O_1 y$ and

$$f^{-1}y_1 \subseteq O_{1/k}(f^{-1}y').$$

Therefore

$$\rho(f^{-1}O_1 y, f^{-1}y') < 1/k,$$

i.e.

$$O_{1/k}(f^{-1}O_1 y) \cap f^{-1}y' \neq \Lambda.$$

But

$$O_{1/k}(f^{-1}O_1 y) \subseteq O_{\varepsilon'/2}(O_{\varepsilon'/2}(f^{-1}y)) \subseteq O_{\varepsilon'}(f^{-1}y).$$

Thus

$$\rho(f^{-1}y, f^{-1}y') < \varepsilon'.$$

Hence, by the choice of ε', it follows that

$$f^{-1}y' \subseteq f^{-1}(O'_y) \subseteq O_\varepsilon(f^{-1}y).$$

Then

$$f^{-1}V \subseteq O_{1/k}f^{-1}y' \subseteq O_{1/k}f^{-1}(O'_y) \subseteq O_{1/k}(O_\varepsilon(f^{-1}y)) \subseteq O_{2\varepsilon}f^{-1}y \subseteq f^{-1}O_y.$$

Consequently, $V \subseteq O_y$. This proves that $\lambda_k(O_1 y) \subseteq O_y$.

Thus the sequence $\{\lambda_n\}$ satisfies condition (Φ). But this means that $\{\tilde{\lambda}_n\}$, where λ_n, $n = 1, 2, \ldots, \infty$, is the system formed by the stars (in $\tilde{Y}$) of the elements of λ_n, is a fundamental set of coverings. Consequently, $\tilde{Y}$ is metrizable, by the theorem of (4). The proof is complete.

Second assertion. *If a completely regular space X has a network $S = \{S_\alpha \mid \alpha \in M\}$ of cardinality $\leq \tau$, then X can be condensed onto a completely regular space of weight $\leq \tau$.*

Proof. By Tikhonov's theorem, X may be regarded as embedded in the product of a sufficiently large set of intervals:

$$X \subseteq \prod_{\beta \in L} I_\beta = J_L.$$

For every pair $S_\alpha, S_{\alpha'} \in S$, for which this is possible, choose some sets U, U' open in J_L, satisfying the conditions: 1) $U \supseteq S_\alpha$, $U' \supseteq S_{\alpha'}$; 2) $U \cap U' = \Lambda$; 3) there exists a finite $L(\alpha, \alpha') \subseteq L$ such that, together with any point $x = \{x_\beta \mid \beta \in L\}$ from the set $U\,(U')$, the set $U\,(U')$ also contains every point $x' = \{x'_\beta \mid \beta \in L\} \in J_L$ for which $x'_\beta = x_\beta$ for all $\beta \in L(\alpha, \alpha')$*. Put

$$K = \bigcup_{\alpha, \alpha' \in M} L(\alpha, \alpha').$$

The cardinality of K does not exceed the cardinality of M, i.e. τ. Let $f : J_L \to J_K$ be the projection and $Y = fX \subseteq J_K$. Since

$$\text{weight}\, Y \leq \text{weight}\, J_K \leq \tau,$$

and f is continuous, it remains for us to prove that $f : J_L \to J_K$ sends distinct points of X

* Sets open in J_L satisfying condition 3) will be called canonical.

to distinct points of the space J_K. Let $x_1 \neq x_2$, $x_1, x_2 \in X \subseteq J_L$, $x_1 = \{x^1_\beta\}$, $x_2 = \{x^2_\beta\}$. Let U_1, U_2 be any canonical sets for which $U_1 \ni x_1$, $U_2 \ni x_2$, and $U_1 \cap U_2 = \Lambda$. By the definition of the net, in S there are $s_{\alpha^1}, s_{\alpha^2}$ such that $x_1 \in s_{\alpha^1} \subseteq U_1$, $x_2 \in s_{\alpha^2} \subseteq U_2$. The pair $s_{\alpha^1}, s_{\alpha^2}$ is marked*, and therefore $L(\alpha, \alpha') \subseteq K$ is defined for it. From condition 3) it follows that there exists $\alpha'' \in L(\alpha, \alpha')$ for which $x^1_{\alpha''} \neq x^2_{\alpha''}$. Since f is the projection and $\alpha'' \in K$, this means that $fx_1 \neq fx_2$. The proof is complete.

Corollary 1. (The addition theorem for weight.) If $X = \bigcup_{\alpha \in M} X_\alpha$, where X is a bicompactum, the cardinality of $M \leq \tau$, and $\sup \text{weight}\, X_\alpha \leq \tau$, then $\text{weight}\, X \leq \tau$. (For another proof see (2).)

Corollary 2. In order that a space which is a closed image of a metric space be metrizable, it is sufficient (and necessary) that it be a paracompact p-space (for another proof see (1)).

Indeed, in (1) it is proved that if a paracompact p-space can be compactified to a metric one, then it is itself metrizable. We note that the assertion of Corollary 2 is meaningful already for bicompacta.

Corollary 3. Let $f : X \to Y$ be a closed mapping of a metric space X onto a locally connected peripherally bicompact space Y. Then Y is metrizable.

This is a new fact. It follows from our main assertion and the corresponding theorem of V. Proizvolov on compactifications (see (7)).

Question: is the ability to be compactified to metric spaces preserved under closed mappings?

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* We shall call a pair $s_{\alpha^1}, s_{\alpha^2} \in S$ marked if there exist sets U, U' satisfying the conditions listed above.

Note: Figure translations are in progress. See original paper for figures.