GENERALIZED DERIVATIVES AND DIFFERENTIABILITY ALMOST EVERYWHERE

MATHEMATICS

1966

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Abstract

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UDC 517.514

MATHEMATICS

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GENERALIZED DERIVATIVES AND DIFFERENTIABILITY ALMOST EVERYWHERE

(Presented by Academician A. D. Aleksandrov, January 17, 1966)

Let G be an arbitrary open domain of n-dimensional Euclidean space R^n. By the symbol $W_p^l(G)$, where $l \geq 1$ is an integer, $p \geq 1$, we shall denote the set of all real functions that are defined and locally integrable in the domain G and such that, in the sense of the theory of generalized functions, all their partial derivatives of order l, in the case $p = 1$, are completely additive set functions defined on the σ-ring of Borel sets contained in G, while for $p > 1$ these derivatives are ordinary functions locally integrable in G to the power p.

Further, B denotes the ball $\{x \in R^n : |x| < 1\}$; C is the set of all functions that are defined and uniformly continuous in the ball B; L_p is the set of all functions integrable in B to the power p, where $p > 1$. The norm in the spaces C and L_p is defined in the usual way. By the symbol L_1 we shall denote the totality of all completely additive functions defined on the σ-ring of Borel sets contained in the ball B. We define the norm in L_1 by putting, for $\varphi \in L_1$, $\|\varphi\|_{L_1} = |\varphi|(B)$, where $|\varphi|(E)$ is the absolute variation of the set function $\varphi(E)$.

Let $\mathfrak{R}$ be an arbitrary Banach space whose elements are real functions defined on the ball B; let u be an arbitrary measurable function defined in the domain G; let x be a point of the domain; and let $P_x(X)$ be a polynomial in the variable $X = (X_1, X_2, \dots, X_n)$ of degree not exceeding l. We shall say that $P_x(X)$ is the complete differential of order l of the function u at the point x in the sense of convergence in $\mathfrak{R}$, if

$$\left\| \frac{u(x + hX) - P_x(hX)}{h^l} \right\|_{\mathfrak{R}} \to 0$$

as $h \to 0$.

We introduce the following notation. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where α_i, $i = 1, 2, \dots, n$, are nonnegative integers. We put:

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n, \qquad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!.$$

If $X = (X_1, X_2, \dots, X_n)$, then we put $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$. Finally, D^α denotes the differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Let $u \in W_p^l(G)$. The formal differential of order l of the function u at the point $x \in G$ is the polynomial

$$\sum_{0 \le |\alpha| \le l} \frac{D^\alpha u(x)}{\alpha!} X^\alpha. \tag{1}$$

In the case $p = 1$, the derivative $D^\alpha u$, where $|\alpha| = l$, is a completely additive set function in G. In this case $D^\alpha u(x)$ in formula (1) should be understood as the derivative of the set function $D^\alpha u$ with respect to Lebesgue measure in R^n.

In what follows W_p^l denotes the space $W_p^l(B)$. We introduce a norm in W_p^l, setting, for $u \in W_p^l$,

$$\|u\|_{W_p^l} = \sum_{0 \le |\alpha| \le l} \|D^\alpha u\|_{L_p}.$$

The main result of the present paper is the following theorem.

Theorem 1. *For every function $u \in W_p^l(G)$, its formal differential of order l at the point $x \in G$ is a complete differential of order l at the point x in the sense of convergence in W_p^l for almost all $x \in G$.*

In other words, if $u \in W_p^l(G)$, then for almost all $x \in G$ the equality

$$\lim_{h \to 0} \left\| \frac{1}{h^l} \left[u(x + hX) - \sum_{0 \le |\alpha| \le l} \frac{D^\alpha u(x)}{\alpha!} h^{|\alpha|} X^\alpha \right] \right\|_{W_p^l} = 0 \tag{2}$$

holds.

The proof of the theorem is based on the following considerations. Denote the expression under the norm sign in equality (2) by $R_x(h, X)$. For $|\alpha| = l$, obviously, we have

$$D_X^\alpha R_x(h, X) = D^\alpha u(x + hX) - D^\alpha u(x). \tag{3}$$

If v is a function locally integrable in G to the power $p \ge 1$, then, as is known (see, for example, (2)), for almost all $x \in G$

$$\|v(x+hX)-v(x)\|_{L_p} \to 0$$

as $h \to 0$. Hence, by virtue of equality (3), it follows that for almost all $x \in G$

$$\sum_{|\alpha|=l} \|D_X^\alpha R_x(h,X)\|_{L_p} \to 0$$

as $h \to 0$.

Let $\varphi_\alpha(X)$, where $|\alpha| \le l-1$, be finite in B infinitely differentiable functions such that

$$\int_B \varphi_\alpha(X) X^\beta \, dX = 0 \quad \text{for } \alpha \ne \beta; \qquad \int_B \varphi_\alpha(X) X^\beta \, dX = 1 \quad \text{for } \alpha = \beta.$$

By virtue of the well-known theorems of S. L. Sobolev on equivalent norms in W_p^l, in order to complete the proof of the theorem it is sufficient to show that, for almost all $x \in G$,

$$\int_B \varphi_\alpha(X) R_x(h,X) \, dX \to 0$$

as $h \to 0$. The validity of this assertion follows from the property of functions locally integrable to the power $p \ge 1$ indicated above, and also from the equality

$$\int_B \varphi_\alpha(X) R_x(h,X) \, dX = \frac{1}{h^l} \int_0^h t^{l-1} \left[\int_B \sum_{|\beta|=l} \frac{D^\beta u(x+tX) - D^\beta u(x)}{\beta!} X^\beta \varphi_\alpha(X) \, dX \right] dt,$$

valid for almost all $x \in G$.

As one of the consequences of Theorem 1, we note the following result.

Theorem 2. Let $u \in W_p^l(G)$, where $lp > n$. Then for almost all $x \in G$ the equality

$$u(x+X) - \sum_{0 \le |\alpha| \le l} \frac{D^\alpha u(x)}{\alpha!} X^\alpha = o(|X|^l). \tag{4}$$

holds.

Proof. By Theorem 1, for almost all $x \in G$,

$$\left\| \frac{1}{h^l} \left[u(x+hX) - \sum_{0\le|\alpha|\le l} \frac{D^\alpha u(x)}{\alpha!} h^{|\alpha|} X^\alpha \right] \right\|_{W_p^l} \to 0$$

as $h \to 0$. By the embedding theorem for the class W_p^l in C when $lp > n$ (^{1}), it follows that, for almost all $x \in G$,

$$\left\| \frac{1}{h^l} \left[u(x+hX) - \sum_{0\le|\alpha|\le l} \frac{D^\alpha u(x)}{\alpha!} h^{|\alpha|} X^\alpha \right] \right\|_{C} \to 0$$

as $h \to 0$, which, as is not difficult to see, is equivalent to equality (4). For the particular case $l = 1$, the theorem was proved earlier by A. Calderón (^{3}) (see also (^{4})).

In conclusion we note that, in an analogous way, by combining Theorem 1 with embedding theorems, one can obtain a number of known results on differentiability almost everywhere for certain classes of functions, for example the theorem on the existence almost everywhere of the second differential of a convex function (^{5}), the theorem on differentiability of a monotone function of the class W_p^1, where $p > h - 1$ (^{6}), and others.

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Received
27 XII 1965

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Note: Figure translations are in progress. See original paper for figures.

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