THE STRUCTURE OF DEPRECIATION CHARGES UNDER A STATIONARY LOAD OF A MACHINE FLEET

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Abstract

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THE STRUCTURE OF DEPRECIATION CHARGES UNDER A STATIONARY LOAD OF A MACHINE FLEET

In this note the principle for constructing depreciation charges in a stationary operating regime of a machine fleet, outlined in (^{1}), is extended to the case of an arbitrary stationary load. After a general description of the model, the solution is constructed for the problem of a fleet with the least expensive structure, and then a system of value indicators, including depreciation charges, corresponding to this optimal solution.

The model studied in the article includes a fleet consisting of machines of different degrees of wear, and a "customer" who determines the load that the fleet is to carry. The conventionality we have adopted makes it possible to combine the nonuniformity and the requirement of stationarity of this load.

The size of the fleet (the number of machines) is taken to be unity. The structure of the fleet is characterized by the distribution of machines according to their (remaining) resource, which varies from 0 to $\overline{R}$. Thus, by the structure of the fleet we shall mean a probability measure μ specified on $\mathbf{R} = [0, \overline{R}]$, or the corresponding distribution function $G_\mu(R)$ ($G_\mu(R)$ is the share of machines whose resource is less than R).

Let the load in the system vary from 0 to 1, and let on $\mathbf{E} = [0, 1]$ there be given a left-continuous nonincreasing function $F(x)$ ($F(0) = 1$, $F(1 + 0) = 0$), such that in any time interval $[t, t + \Delta]$ the duration of a load not less than x (measured in fractions of the fleet) is equal to $F(x)\Delta$.

In what follows, $F(x)$ will be called the **structure** of the (stationary) **load**.

To carry this load, **assignments** are made among machines with different resources. Let $\psi(\rho, \xi)$ be the share of machines whose resource lies in the interval $\rho \subset \mathbf{R}$ and the assignments for which lie in the interval $\xi \subset \mathbf{E}$. Here by an assignment for a machine is meant the share of time during which the machine is under load (on an infinitely small interval $[t, t + \Delta]$). If a machine with resource R performs assignment x during time Δ, then its resource decreases by $x\Delta$.

Machines with zero resource are not used and are replaced by new machines with resource $\overline{R}$.

Let ψ be a nonnegative function of intervals, additive in each of its variables, satisfying for all $\rho \subset \mathbf{R}$ the condition

$$\psi(\rho, \mathbf{E}) = \mu(\rho). \tag{1}$$

Put

$$\psi(\mathbf{R}, \xi) = \nu(\xi). \tag{2}$$

It is clear that $\nu(\cdot)$ is a probability measure on $\mathbf{E}$. It is associated with the distribution function $\Psi_\nu(x)$. We shall call this function the **structure of assignments** performed under their given assignment ψ.

The problem of selecting a fleet of some age structure μ that performs a load of structure F can be divided into two parts: 1) the choice of the best structure of assignments Ψ and of a method for realizing the load F with the aid of

of this structure of jobs and 2) the choice of the fleet and the best allocation of jobs for their fixed structure. In this formulation, relation (2) becomes an additional condition that the function ψ must satisfy. We shall begin with this "internal" problem of the best selection of the fleet and allocation of jobs.

Define the function $\psi(\rho, \xi)$ as follows. Consider the rectangle $R \times E$. For each $\alpha \in [0, 1]$, let

$$R_\alpha = \sup\{R \mid \mu[0, R] \le \alpha\}, \qquad x_\alpha = \sup\{x \mid \nu[0, x] \le \alpha\}. \tag{3}$$

Set

$$\psi(\rho, \xi) = \operatorname{mes}\{\alpha \mid R_\alpha \in \rho,\ x_\alpha \in \xi\}. \tag{4}$$

The collection of pairs $\{R_\alpha, x_\alpha\}$ defines a nondecreasing assignment-function $x(R)$ as a function of the machine resource. This function is determined uniquely $\operatorname{mod}\mu$.

The main interest in the study of such systems is the stationary state of the system. Such a state is defined as a triple $\langle \mu, \psi, \nu \rangle$ satisfying the conditions

$$\psi(\rho, E) = \mu(\rho), \tag{5}$$

$$\psi(R, \xi) = \nu(\xi), \tag{6}$$

$$\int_0^1 x\psi(\rho, dx) = a\mu(\rho), \tag{7}$$

where a is the rate of wear (and purchase) of machines,

$$a = \int_0^1 x\nu(dx). \tag{8}$$

Let K_ν be the set of all stationary states with job structure ν. Among these states there is a unique state $\langle \tilde{\mu}_\nu, \tilde{\psi}_\nu, \nu \rangle$, in which $\tilde{\psi}$ is determined from condition (4). (It can be constructed in the following way: $x(R)$ is determined from $\nu(\alpha)$, and then $\tilde{\mu}$ and $\tilde{\psi}$ are determined from $x(R)$.)

Theorem 1. Whatever the stationary state $\langle \mu, \psi, \nu \rangle$ from K_ν, for every $R \subseteq R$

$$\mu[0, R] \leq \tilde{\mu}_\nu[0, R]. \tag{9}$$

Corollary. Whatever the monotonically decreasing function $c(R)$ of the cost of machines of resource R, the cheapest fleet of stationary structure for the job assignment ν is $\tilde{\mu}_\nu$.

We now turn to the choice of the best job structure. First of all, it is necessary to determine the set of those job structures that can realize a given load structure.

Call a function $w(\tau, \xi)$ of two intervals $\tau \subset E$ and $\xi \subset E$ a *work order* if it satisfies the following condition: there exists a set $A \subset E \times E$ such that

$$w(\tau, \xi) = \operatorname{mes} A \cap (\tau \times \xi), \qquad \tau \subset E, \quad \xi \subset E. \tag{10}$$

We shall say that the load structure F is *realizable* by the job structure ν if there exists a work order w such that, for any x,

$$w(E, [0, x]) = \Psi_\nu^{-1}(x) \tag{11}$$

and for any t

$$w([0, t], E) = F^{-1}(t). \tag{12}$$

Theorem 2. In order that the load structure F be realizable by the assignment structure ν, it is necessary and sufficient that, for every x,

$$\int_0^x F(z)\, dz \leqslant \int_0^x \Psi_\nu^{-1}(z)\, dz. \tag{13}$$

A special role in our consideration is played by one of the extreme points of the set of assignment structures—the assignment structure $\tilde{\nu}$, for which

$$\Psi_{\tilde{\nu}}^{-1}(x) = F(x). \tag{14}$$

To this assignment structure (and to it alone) there corresponds the following rule for the distribution of loads: for each residual resource of machines R, the system load level $x(R)$ is specified, and the machine is used at all loads of this and higher levels. This function is nonincreasing and $x(+0) = 1,\ x(\overline{R}) = 0$. It is given by the relation

$$\int_{x(R)}^{1} F(x)\,dx = dR/\overline{R}. \tag{15}$$

Theorem 3. Whatever the assignment structure ν realizing F, for any $R \in \mathbf{R}$

$$\mu_\nu[0, R] \leqslant \tilde{\mu}_{\tilde{\nu}}[0, R]. \tag{16}$$

Our problem will now be to obtain, mutually related, the valuation $\eta(R)$ of a machine with resource R, the payment $c(x)$ for the operation of one machine under assignment x, and the depreciation deductions per unit of working time $A_W(R)$ and per unit of calendar time $A_L(R)$, which stimulate the use of the optimal regime $\tilde{\nu}$. Naturally, in addition to the structure, the cost of a new machine K and the efficiency norm α must enter into the definition of these indicators.

Let us denote the lifetime of a machine with resource R, when the regime $\tilde{\nu}$ is used, by $t(R)$, and the inverse function by $r(t)$. We have

$$t(R) = \int_0^R \frac{dr}{F(x(r))}. \tag{17}$$

Put

$$A_W(R) = \kappa e^{-\alpha t(R)}, \tag{18}$$

$$c(x) = \kappa e^{-t(x^{-1}(x))}, \tag{19}$$

$$\eta(R) = -\int_0^{t(R)} e^{-\alpha t} \int_{x(r(t))}^{1} c(x)\,dF(x)\,dt, \tag{20}$$

$$A_L(R) = \alpha\eta(R). \tag{21}$$

The meaning of equalities (20) and (21) is simple: $\eta(R)$ is determined from the condition of recoupment of the machine under the regime $\tilde{\nu}$; the depreciation deduction over the lifetime $A_L(R)$ is a compensation for the tying-up of capital. Equality (18) expresses the basic principle of the proposed method for calculating depreciation deductions: the discounted deductions for the working time of a given machine are the same at all moments of time and, in particular, are equal to the discounted payment for work at the maximum load in the last moment of its operation (this interpretation of $\varkappa$ follows from (19) for $x = 1$). $c(x)$ is determined from the usual marginal relations: the value of operation must be such as to compensate the depreciation charges A_W at the threshold load $x(R)$.

The quantity $\varkappa$ is determined from the condition $\eta(\bar{R}) = K$

$$\varkappa = \alpha K \left(1 + \int_0^1 e^{-\alpha t(R(x))} \, dF(x) \right)^{-1} . \tag{22}$$

If, in the expression for $\eta(R)$, we interchange the order of integration and use the expression for $c(x)$, we obtain

$$A_L(R) = \alpha\eta(R) = -\int_{x(R)}^1 dF(x) \, [c(x) - A_W(x)], \tag{23}$$

i.e., the depreciation charge for calendar time is equal to the difference between the integral payment for operation and the depreciation charges for the working time of the machine spent in this operation. We note that the same structure of valuations is encountered for other reasons in the theory of differential rent (see K. Marx, *Capital*, vol. 3) and in problems of inventory theory ([2]).

To each load structure v there corresponds a certain payment for all the operation of one machine $C(v)$, and the depreciation charges for this machine $A(v)$.

Theorem 4. Whatever the load structure v,

$$A(v) \geq C(v) \tag{24}$$

and for $v = \tilde{v}$

$$A(v) = C(v). \tag{25}$$

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CITED LITERATURE

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2. T. C. Koopmans, *Proc. I Intern. Conf. on Operational Research*, 1957.

Note: Figure translations are in progress. See original paper for figures.

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