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Abstract

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NECESSARY CONDITIONS FOR AN EXTREMUM AND THEIR APPLICATION TO THE STUDY OF CERTAIN EQUATIONS

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Let X be a Banach space, $\Omega \subset X$. An element $u \in X$ will be called an admissible direction for the set Ω at the point $x \in \overline{\Omega}$ if there exist a sequence $u_s \in X$ and a numerical sequence α_s such that 1) $x + \alpha_s u_s \in \Omega$, 2) $u_s \to u$, 3) $\alpha_s > 0$, $\alpha_s \to 0$. The admissible directions for the set Ω at the point x form a closed cone, which we shall denote by $M_x(\Omega)$. Note that always $0 \in M_x(\Omega)$.

Let us describe the cone $M_x(\Omega)$ in several important special cases.

1. Let φ be a functional defined and continuous in some neighborhood $V(x_0)$ of the point $x_0 \in X$, and let $\Omega = \{x \in V(x_0) \mid \varphi(x) = \varphi(x_0)\}$. Then

$$M_{x_0}(\Omega) = \{u \in X \mid \operatorname{grad} \varphi(x_0)(u) = 0\}.$$

2. Let φ be the functional considered above,

$$\Omega = \{x \in V(x_0) \mid \varphi(x) \le \varphi(x_0)\}.$$

   In this case

$$M_{x_0}(\Omega) = \{u \in X \mid \operatorname{grad} \varphi(x_0)(u) \le 0\}.$$

3. Suppose that Ω is a convex set, $x \in \overline{\Omega}$. In this case $M_x(\Omega)$ is the closed conical hull of the set $\Omega - x$.
4. Let X be a finite-dimensional Banach space, Ω' a convex solid polyhedron in X, Ω the boundary of Ω', and x_0 a vertex of Ω'. It is easy to see that $M_{x_0}(\Omega)$ is the boundary of the closed conical hull of the set $\Omega' - x_0$. Note that in this case $M_{x_0}(\Omega)$ is a nonconvex cone.

Consider a functional f defined on the space X. If f is differentiable at some point $x \in X$, we put $Fx = \operatorname{grad} f(x)$.

We indicate necessary conditions for a minimum of the functional f on the set Ω. Here by a minimum we shall everywhere mean a local minimum.

Theorem 1. *Let the functional f be defined on the set Ω, attain a minimum there at the point y, and be Fréchet differentiable at this point. Then*

$$\min_{u\in M_y(\Omega)} Fy(u) = 0.$$

We shall call the set Ω pseudoconvex if, for any $x \in \Omega$, the following conditions are satisfied: a) if $h \in X^*$ is such that $h(M_x(\Omega)) \geq 0$ and for some $u \in M_x(\Omega)$ $h(u) > 0$, then $h(\Omega - x) \geq 0$; b) if $h \in X^*$ is such that $h(M_x(\Omega)) = 0$, then either $h(\Omega - x) \geq 0$, or $h(\Omega - x) \leq 0$.

It is clear that every convex set is pseudoconvex. An example of a pseudoconvex but nonconvex set is the boundary of a convex solid set.

Theorem 2. *If the functional f, defined on a pseudoconvex set Ω, attains a minimum there at the point y and is Fréchet differentiable at this point, then either*

$$\min_{x\in\Omega} Fy(x - y) = 0,$$

or

$$\max_{x\in\Omega} Fy(x - y) = 0.$$

Theorem $2'$ (see ([1])). *If the functional f, defined on a convex set Ω, attains a minimum there at the point y and is Gâteaux differentiable at this point, then*

$$\min_{x\in\Omega} Fy(x - y) = 0.$$

Let $\Gamma \subset X^*$. In what follows we shall consider sets Ω satisfying the following condition:

$(*)$ If $h \in \Gamma$, then there exist unique elements y_h and z_h such that

$$h(y_h) = \min_{x\in\Omega} h(x), \tag{1}$$

$$h(z_h) = \max_{x\in\Omega} h(x). \tag{2}$$

Let Ω satisfy condition $(*)$ with respect to Γ. Consider the operators G_Ω and H_Ω, acting from Γ into Ω, as follows: $G_\Omega h = y_h$, $H_\Omega h = z_h$ (here y_h and z_h are defined, respectively, by formulas (1) and (2)).

We give an example of the operators G_Ω and H_Ω. Let H be a Hilbert space; X a Banach space; B a linear bounded operator acting from H into X, whose range is dense in X; $\Gamma = X^* \setminus \{0\}$, $\Omega = \{x \in X \mid x = Bz,\ \|z\| = 1\}$. In this case, for $h \in \Gamma$,
$G_\Omega h = -BB^*h/\|B^*h\|$, $H_\Omega h = BB^*h/\|B^*h\|$.

Let Ω be some set on which a differentiable functional f is defined. Put $\Gamma_f = \{h \in X^* \mid h = F'x,\ x \in \Omega\}$.

Theorem 3. Let the functional f be defined and differentiable in the Fréchet sense on a pseudoconvex set Ω, which has property $(*)$ with respect to Γ_f, and attain a minimum on Ω at a point y. Then y satisfies one of the two equations $x = H_\Omega F'x$ or $x = G_\Omega F'x$.

Theorem 3′. Let the functional f be defined and differentiable in the Gâteaux sense on a convex set Ω, which has property $(*)$ with respect to Γ_f, and attain a minimum on Ω at a point y. Then y satisfies the equation $x = G_\Omega F'x$.

We note that all the theorems formulated above carry over, with obvious modifications, to the case when a maximum is considered instead of a minimum.

Theorem 4. Let a strongly potential operator F* (the gradient of a functional f) be defined on a pseudoconvex set Ω, which has property $(*)$ with respect to Γ_f. Suppose further that one of the following conditions is satisfied: a) Ω is compact; b) f is weakly lower or upper semicontinuous, Ω is weakly compact. Then one of the equations $x = G_\Omega Fx$ or $x = H_\Omega Fx$ has a solution.

Theorem 4′. Let a potential operator F (the gradient of a functional f) be given on a convex set Ω, which has property $(*)$ with respect to Γ_f. Suppose further that one of the following conditions is satisfied: a) f is a continuous functional, Ω is compact; b) f is a weakly continuous functional, Ω is weakly compact. Then both equations $x = G_\Omega Fx$ and $x = H_\Omega Fx$ have solutions.

If in condition b) one requires only weak lower (upper) semicontinuity of f, then one can guarantee only the existence of solutions of the equation $x = G_\Omega Fx$ (respectively, $x = H_\Omega Fx$).

We give one consequence of Theorems 4 and 4′.

Theorem 5. Let H be a Hilbert space; X a Banach space; B a linear bounded operator acting from H into X, whose range is dense in X. Suppose further that F is a strongly potential operator defined on the set $\Omega = \{x \in X \mid x = Bz,\ \|z\| = R\}$, and $Fx \neq 0$ $(x \in \Omega)$. Assume that one of the following conditions is satisfied: a) B is a completely continuous operator; b) F is the gradient of a weakly lower or upper semicontinuous functional. Then there exists a number λ_0 such that the equation $x = \lambda_0 BB^* Fx$ has at least

* An operator F is called potential (strongly potential) if it is the Gâteaux (Fréchet) derivative of some functional f (see (2)).

one solution of the form $x_0 = Bz_0$ $(\|z_0\| = R)$. In this case either $\lambda_0 = R/\|B^* Fx_0\|$, or $\lambda_0 = -R/\|B^* Fx_0\|$.

Theorem 5′. Let H and X be the same spaces, and B the same operator, as in Theorem 5; let F be a potential operator defined on the set $\Omega = \{x \in X \mid x = Bz,\ \|z\| \leq R\}$, with $Fx \neq 0$ $(x \in \Omega)$. Suppose that one of the following conditions is satisfied: a) F is the gradient of a continuous functional, and B is a completely continuous operator; b) F is the gradient of a weakly continuous functional. Then for any $0 < r \leq R$ there exist numbers λ_1 and λ_2 such that, for $i = 1, 2$, the equations $x = \lambda_i BB^* Fx$ have at least one solution of the form $x_i = Bz_i$ ($\|z_i\| = r$). In this case $\lambda_1 = -r/\|B^* Fx_1\|$, $\lambda_2 = r/\|B^* Fx_2\|$. If in condition b) F is the gradient of a functional weakly lower (upper) semicontinuous, then there exists λ_0 such that the equation $x = \lambda_0 BB^* Fx$ has at least one solution of the form $x_0 = Bz_0$ ($\|z_0\| = r$). In this case, from weak lower (upper) semicontinuity it follows that $\lambda_0 = -r/\|B^* Fx_0\|$ ($\lambda_0 = r/\|B^* Fx_0\|$).

If F is the gradient of a convex (concave) functional, then there exists a unique negative (positive) number λ_0 such that the equation $x = \lambda_0 BB^* Fx$ has a solution of the form $x_0 = Bz_0$ ($\|z_0\| = r$), and this solution is unique.

Theorems 5 and 5′ are a generalization of Theorems 15.1–15.4 in ([2]).

Let us note in conclusion that, for solving the equations $x = G_\Omega Fx$ and $x = H_\Omega Fx$ in the case where Ω is a convex set, the method of successive approximations described in ([1]) may be applied.

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[2] M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.

Note: Figure translations are in progress. See original paper for figures.