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Abstract

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MECHANICS

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SETTLING OF A CLOUD OF INTERACTING PARTICLES AND THE FORMATION, IN THIS PROCESS, OF A "DUSTING" SOURCE AS A RESULT OF THE ACTION OF ATMOSPHERIC DIFFUSION

(Presented by Academician E. K. Fedorov, 9 VI 1965)

The article investigates the distribution of concentration on the Earth' s surface in the case of settling of a cloud of interacting particles at concentration $q > q_0$, formed as a result of the action of a point source (q_0 is a certain constant subject to experimental determination). We do not here go into the causes that produce this interaction—they may in each case be the subject of a special study. Owing to turbulent diffusion occurring in time and space (and possibly also for other reasons), the cloud becomes diluted only from its surface; in this process the concentration of particles in its diluted part becomes less than q_0. In this part of the cloud the interaction of the particles is disrupted, and the particles separated from the cloud fall with the Stokes velocity w. It is assumed that the cloud as a whole settles with constant velocity V.

The particles separated from the cloud form a "dusting" source. The source intensity is $Q = Q_0 f(t)\delta(x)\delta(y)\delta[z - h(1 - t/T) \times U(1 - t/T)]$, where $U(t)$ is the unit function; $T = h/V$; $f(t) = \beta t e^{-\alpha t}$; Q is the number of particles in the source; h is the height of the source location; xy is the plane of the Earth; z is the vertical coordinate; t is time.

The function Q determines the character of the change in source intensity depending on its position in space and in time. The quantity β is determined from the condition

$$\beta \int_0^\infty \tau e^{-\alpha\tau}\, d\tau = 1, \qquad \beta = \alpha^2.$$

The number of particles diffusing from the cloud during the time T is determined from the expression

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$$Q_0\alpha^2 \int_0^T \tau e^{-\alpha\tau}\, d\tau = Q_0 \left[1 - \alpha e^{-\alpha T}\left(T + \frac{1}{\alpha}\right)\right].$$

The function $f(t)$ is chosen so that at some point of the time interval $(0, \infty)$ it has a maximum and at the ends of this interval is equal to zero.

The equation of turbulent diffusion of a heavy impurity is solved:

$$\partial q/\partial t = K_x \partial^2 q/\partial x^2 + K_y \partial^2 q/\partial y^2 + K_z \partial^2 q/\partial z^2 - u\, \partial q/\partial x + w\, \partial q/\partial z + Q. \quad (1)$$

Here u is the horizontal component of the wind velocity; w is the gravitational settling velocity of the impurity; K_x, K_y, K_z are the corresponding coefficients of turbulent diffusion of the impurity.

Equation (1) is solved under the following initial and boundary conditions: at $t = 0$, $q = 0$; at $z = 0$, $q = 0$; as $\sqrt{x^2 + y^2 + z^2} \to \infty$, $q = 0$.

All coefficients of equation (1) are assumed constant. In such-

in which case its solution is written in the following form:

$$q = \frac{Q\alpha^2}{8\left(\pi^3 K_x K_y K_z\right)^{1/2}} \int_0^t F(t, \tau, x, y, z)\, d\tau \quad \text{for } t \leqslant T; \quad (2a)$$

$$q = \frac{Q\alpha^2}{8\left(\pi^3 K_x K_y K_z\right)^{1/2}} \int_0^T F(t, \tau, x, y, z)\, d\tau + \\ + \frac{Q'\alpha^2}{8\left(\pi^3 K_x K_y\right)^{1/2} K_z^{3/2}} \int_T^t \frac{z f(\tau)}{(t-\tau)^{5/2}} \exp\left[-\frac{[x - u(t-\tau)]^2}{4K_x(t-\tau)} - \frac{y^2}{4K_y(t-\tau)} - \\ - \frac{w^2(t-\tau)}{4K_z} - \frac{wz}{2K_z} - \frac{z^2}{4K_z(t-\tau)}\right] d\tau \quad \text{for } t \geqslant T, \quad (2b)$$

where

$$Q' = \lim_{h \to 0} (Q_0 h),$$

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$$F(t,\tau,x,y,z) = \frac{f(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{[x-u(t-\tau)]^2}{4K_x(t-\tau)} - \frac{y^2}{4K_y(t-\tau)} - \right.$$
$$\left. -\frac{w^2(t-\tau)}{4K_z} - \frac{w\left[z-h\left(1-T^{-1}\tau\right)\right]}{2K_z}\right] \left\{\exp\left[-\frac{\left[z-h\left(1-T^{-1}\tau\right)\right]^2}{4K_z(t-\tau)}\right]\right.$$
$$\left. -\exp\left[-\frac{\left[z+h\left(1-T^{-1}\tau\right)\right]^2}{4K_z(t-\tau)}\right]\right\}.$$

The second term in expression (2b) tends to zero as a result of the condition of absorption of the impurity on the surface $z = 0$. The obtained solution is valid for an arbitrary specification of $h(t)$ as a function of time. In this formulation of the problem, the concentration distribution on the Earth' s surface has three extremal points (two maxima and one minimum).

The paper gives simple formulas for computing the coefficients of turbulent scattering from experimentally measured distances from the point of impurity release to the points corresponding to the minimum and to the diffusion maximum of the surface concentration (in the direction of the wind).

Let us make in (2a) and (2b) the change of variable $t - \tau = \xi$; then

$$q = \int_0^t F(t,\xi,x,y,z)\,d\xi, \qquad t \leqslant T; \tag{3a}$$

$$q = \int_{t-T}^t F(t,\xi,x,y,z)\,d\xi, \qquad t \geqslant T. \tag{3b}$$

To determine the concentration of particles on the Earth' s surface, q^*, we compute the integral

$$q^* = \int_0^\infty K_z \left.\frac{\partial q}{\partial z}\right|_{z=0} dt. \tag{4}$$

Since the function $q(z,t)$ is defined in the interval $0 \leqslant z \leqslant \infty$, and the integrand, as a function of z and ξ, is continuous in the semi-infinite strip $0 \leqslant \xi \leqslant t$ and has a partial derivative with respect to z in this region, for any z in the interval $[0,\infty]$ the formula holds

$$q^* = H\left\{\int_0^\infty \exp(-C)\int_0^t F_1(\xi,t)\,d\xi\,dt - \int_T^\infty \exp(-C)\int_0^{t-T} F_1(\xi,t)\,d\xi\,dt\right\}, \tag{5}$$

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where

$$F_1(\xi,t)=\frac{t-\xi-T^{-1}(t-\xi)^2}{\xi^{5/2}}\exp\left[-\frac{A}{\xi}-B\xi\right],$$

$$H=\frac{Q\alpha^2h}{8\left(\pi^3K_xK_yK_z\right)^{1/2}},\qquad A=\frac{x^2}{4K_x}+\frac{y^2}{4K_y}+\frac{h^2\left(1-T^{-1}t\right)^2}{4K_z},$$

$$B=\frac{u^2}{4K_x}+\frac{(w-V)^2}{4K_z}-\alpha,\qquad C=-\frac{xu}{2K_x}-\frac{h(w-V)\left(1-T^{-1}t\right)}{2K_z}+\alpha t.$$

Let us compute the inner integral in (5). According to (1), it is equal to

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{-\beta/x}\,dx=\beta^{(\nu-1)/2}u^{(2\mu+\nu-1)/2}\exp\left(\frac{\beta}{2u}\right)\Gamma(\mu)W_{(1-2\mu-\nu)/2,\,\nu/2}\left(\frac{\beta}{u}\right).$$

Expanding $\exp(-B\xi)$ in a Taylor series, after integration we shall have

$$q^*=H\int_0^\infty\exp(-C)\exp\left(-\frac{A}{2t}\right)A^{-3/4}t^{5/4}$$

$$\sum_{n=0}^\infty\frac{(-\beta)^n}{n!}(At)^{n/2}\left[\Gamma(2)W_{-(5/4+n/2),\,(1/4+n/2)}\left(\frac{A}{t}\right)-T^{-1}tW_{-(9/4+n/2),\,(-1/4+n/2)}\left(\frac{A}{t}\right)\right]dt,$$

where $W_{\lambda,\mu}$ is the Whittaker function; Γ is the gamma function.

It is known that the Whittaker function for large values of the argument has the form

$$W_{\lambda,\mu}(z)\sim e^{-z/2}z^\lambda\left(1+\sum_{k=1}^\infty\frac{[\mu^2-(\lambda-1/2)^2]\cdots[\mu^2-(\lambda-k+1/2)^2]}{k!\,z^k}\right).\qquad(6)$$

For sufficiently large x, the condition $A/t\gg1$ is fulfilled. In this case, using the asymptotic expansion of the Whittaker function (6), valid for large values of the argument, we obtain

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$$q^* \simeq H\Gamma(2) \int_0^\infty \exp\left(-d - \frac{a}{t} - bt\right) \exp\left[-\frac{Bh^2}{4K_z T^2}(t-T)^2\right] t^{5/2} A^{-2}\, dt - \\ - \int_T^\infty \exp(-C) \int_0^{t-T} F_1(\xi, t)\, d\xi\, dt, \qquad (*)$$

where

$$d = -\frac{xu}{2K_x} - \frac{h(w-V)}{2K_z} - \frac{Uh}{2K_z} + B\left(\frac{x^2}{4K_z} + \frac{y^2}{4K_y}\right), \qquad a = \frac{x^2}{4K_x}$$

$$+\frac{y^2}{4K_y} + \frac{h^2}{4K_z}, \qquad b = \frac{wV}{2K_z} - \frac{V^2}{4K_z} + \alpha.$$

The second exponential under the first integral will be a delta-like function under the condition $Bh^2/4K_z \gg 1$, which is equivalent to the condition $u^2h^2/16K_xK_z \gg 1$. For the free atmosphere ($h > 1000$ m) it is satisfied. The first exponential under the integral is a slowly varying function. Thus one may write

$$q^* \simeq H\Gamma(2)A^{-2} \exp\left[-d - \frac{a}{b} - bT\right] \int_0^\infty \exp\left[-\frac{Bh^2}{4K_z T^2}(t-T)^2\right] dt - \\ - H \int_T^\infty \exp(-C) \int_0^{t-T} F_1(t, \xi)\, d\xi\, dt. \qquad (7)$$

The integral

$$\int_T^\infty \exp(-C) \int_0^{t-T} F_1(t, \xi)\, d\xi\, dt \leq \int_1^\infty \exp(-C) \int_0^{t} F_1(t, \xi)\, d\xi\, dt.$$

Therefore

$$q^* = \gamma(T)\Gamma(2)A^{-2} \exp\left[-d - \frac{a}{T} - bT\right] T^{5/2} \int_0^\infty \exp\left[-\frac{Bh^2}{4K_z T^2}(t-T)^2\right] dt$$

and $1/2 < \gamma(T) \leq 1$ for $0 < T \leq \infty$.

The first term of (7) represents the first term of the expansion of (*) in a Taylor series in a neighborhood of the point $t = T$. Let us estimate the subsequent terms of this expansion. For this purpose we expand $\exp[-a/t - bt]$ in a Taylor series in

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in a neighborhood of the point $t = T$ and integrate the expression

$$\sum_{n=1}^{\infty} \int_0^{\infty} (t-T)^n \exp\left[-\frac{Bh^2}{4K_z T^2}(t-T)^2\right] dt.$$

Taking into account that the time T is a sufficiently large quantity ($T > 10$ sec), we obtain the condition under which all subsequent terms of the expansion will be small:

$$\frac{1}{4}\frac{x^2}{h^2}\frac{V}{u}\sqrt{\frac{K_z}{K_x}} \ll 1.$$

Formula (7) can be used to determine the values of the coefficients K_x and K_z from experimental measurements of the coordinates of the extremal points of the function q^*. Let us determine the extremal points of the function q^*. Equating the derivative $\partial q^*/\partial x$ to zero, we obtain $x_{1,2} = \pm\infty$, and also

$$\tilde{x}^3 - \tilde{x}^2 + \tilde{b}_1\tilde{x} - \tilde{b}_0 = 0, \tag{8}$$

where $\tilde{x} = x(BT+1)/uT$, $\tilde{b}_1 = 8K_x(BT+1)/u^2T + \tilde{y}^2K_x/K_y$, $\tilde{b}_0 = \tilde{y}^2K_x/K_y$, $\tilde{y} = y(BT+1)/uT$.

Putting $y = 0$, we find that $\tilde{b}_0 = 0$,

$$\tilde{x}\left(\tilde{x}^2 - \tilde{x} + \tilde{b}_1\right) = 0, \tag{9}$$

therefore,

$$\tilde{x}_3 = 0, \qquad \tilde{x}_4 = \frac{1}{2}\left(1 - \sqrt{1 - 4\tilde{b}_1}\right), \qquad \tilde{x}_5 = \frac{1}{2}\left(1 + \sqrt{1 - 4\tilde{b}_1}\right). \tag{10}$$

At the points $\tilde{x}_3$ and $\tilde{x}_5$ the function q^* has a maximum, and at the point $\tilde{x}_4$ a minimum. In fact the first maximum is located at the point $x_3 = uT$. This discrepancy in the position of x_3 was obtained because of the condition $A/t \gg 1$ in deriving equation (8).

In the case $\tilde{b}_1 = 1/4$, the roots $\tilde{x}_4$ and $\tilde{x}_5$ coincide. At the point $\tilde{x}_4 = \tilde{x}_5$ the function q^* has an inflection point. The maximum of the function q^*, formed due to diffusion of the particles, disappears. There remains a maximum only at the point $x_3 = uT$. It follows from (10) that, for $\tilde{b}_1 \leq 1/4$, there exist a minimum and a distant maximum of the function q^*.

Under the assumption $y = 0$,

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$$\tilde{b}_1 = 8K_x(BT+1)/u^2T$$

or

$$\tilde{b}_1 = 2 + \frac{2(w-V)^2}{u^2}\frac{K_x}{K_z} + \frac{K_x}{u^2}\left(\frac{1}{T} - \alpha\right) \leq \frac{1}{4}. \qquad (11)$$

If the maximum of the function $f(t)$ is on the surface of the Earth ($\alpha = 1/T$), then condition (11) is not satisfied. The simplest way to determine the quantities K_x and K_z is to use the properties of the roots of the quadratic equation

$$\tilde{x}^2 - \tilde{x} + \tilde{b}_1 = 0,$$

$$x_4 + x_5 = \frac{Tu}{BT+1}, \qquad x_4x_5 = \tilde{b}_1\left(\frac{Tu}{BT+1}\right)^2. \qquad (12)$$

From system (12) we determine the coefficients K_x and K_z:

$$K_x = \frac{x_4x_5u}{8(x_4+x_5)}, \qquad K_z = \frac{1}{4}\,\frac{(w-V)^2}{\alpha - 1/T + u(x_4x_5 - 2(x_4+x_5)^2)/(x_4+x_5)x_4x_5}. \qquad (13)$$

Formulas (13) make it possible to calculate the values of the turbulent-scattering coefficients from measurements of the positions of the extremal points of the function q^*.

Let us examine the following special cases:

a) $T = 0$ ($V \to \infty$). In this case condition (11) is not satisfied.

b) $T \to \infty$ ($V = 0$). In this case there is a source located at height h with an intensity varying in time.

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Note: Figure translations are in progress. See original paper for figures.

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