CONVERGENCE OF VARIATIONAL-DIFFERENCE SCHEMES WITH IMPROVED APPROXIMATION OF THE BOUNDARY

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Abstract

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MATHEMATICS

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CONVERGENCE OF VARIATIONAL-DIFFERENCE SCHEMES WITH IMPROVED APPROXIMATION OF THE BOUNDARY

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In the variational problem corresponding to a linear elliptic equation, the coordinate functions can be chosen in such a special form that the structure of the corresponding algebraic system will coincide with the structure of the difference equations.

In ([1]-[4]) the variational approach was applied to the construction and study of difference schemes for the first and second boundary-value problems for a second-order equation. In ([2]) the advantages of this approach were noted, in particular, for estimating the rate of convergence of difference schemes. The minimum principle makes it possible to reduce the estimate of the rate of convergence of a scheme to the problem of approximating, in the energy norm, the sought solution by functions from the subspace $\mathfrak{M}$ in which the minimum of the functional is sought. In ([3]) a scheme was constructed that converges with rate $O(\sqrt{h})$, where h is the mesh size. The comparatively low order of the convergence rate is due to the fact that, for the sake of uniformity (regularity) in the construction, the functions $\tilde{u} \in \mathfrak{M}$ were taken equal to zero in a boundary strip of width $O(h)$. But by violating regularity one can substantially improve the rate of convergence. This circumstance was used in ([4]) in the study of degenerate equations. However, in violating regularity, it is natural to try to preserve the block structure of the matrices, which permits the use of sweep methods, and to avoid excessively large condition numbers of these matrices.

In the present work we construct variational difference schemes for the third boundary-value problem for a linear two-dimensional elliptic equation of the second order. The established rate of convergence $O(h)$ cannot be improved. The matrix of the system has a block tridiagonal structure. The condition number of the matrix has order h^{-3}. The results obtained are also valid for the first boundary-value problem.

1. Consider the problem*

Fig. 1

Figure 1: Fig. 1

$$(a_{ij}u_{\xi_i})_{\xi_j} - cu = f, \tag{1}$$

$$\left[a_{ij}\cos(\nu,\xi_j)u_{\xi_i} + \sigma u\right]_{\Gamma(\omega)} = 0 \tag{2}$$

in a domain ω of the variables ξ_1, ξ_2 with boundary $\Gamma(\omega)$ of class $C^{(3)}$. Here $f \in L_2$, the function $\sigma \geq 0$, σ' is continuous on $\Gamma(\omega)$, and σ is not identically zero. The coefficients $a_{ij}(\xi_1, \xi_2)$ and $c(\xi_1, \xi_2) \geq 0$ are assumed given in some domain $\omega' \supset \bar{\omega}$. The functions c, $\partial a_{ij}/\partial\xi_1$, $\partial a_{ij}/\partial\xi_2$ are assumed bounded, the matrix $\{a_{ij}\}$ is positive definite, and ν is the outward normal to $\Gamma(\omega)$.

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* The notation adopted in the paper is as follows: capital Latin letters denote constants independent of the mesh size h; lowercase Greek letters used as subscripts denote differentiation; summation from 1 to 2 is performed over repeated indices; the symbol $\| \|_{k,\omega}$ denotes the norm in the space $W_2^k(\omega)$ of S. L. Sobolev.

It is known ([5]) that $\|u\|_{2,\omega} \leq C_1\|f\|_{0,\omega}$; we shall assume that u has been extended ([6]) to ω' so that $\|u\|_{2,\omega'} \leq C_2\|u\|_{2,\omega}$. Let the number A be such that all circles of radius A tangent to $\Gamma(\omega)$ have exactly one common point with $\Gamma(\omega)$. Choose two numbers: a sufficiently large number P, for example $P > 20$, and an arbitrary number G. Using these numbers and the number A, compute the number Q, whose properties are described below. Superimpose on $\Gamma(\omega)$ a square mesh of step h, and draw a contour Γ parallel to $\Gamma(\omega)$ at a distance Qh^2 outside ω. Construct an approximating polygon whose vertices lie at the points of intersection of Γ with the mesh lines and whose side lengths lie between Ph and $(2P+2)h$ (it is easy to give algorithms for such a construction). Then the endpoints of sides lying at a distance less than $\sqrt{2}Gh^2$ from mesh nodes are moved to those nodes. The resulting polygon will be denoted by Ω, and its boundary by $\Gamma(\Omega)$. The number Q must be chosen so that $\omega \subset \Omega$ and so that the distance from any point belonging to $\Gamma(\Omega)$ to ω is less than $2Qh^2$.

Fig. 1

Divide the mesh cells by diagonals inclined at an angle of 45° to the ξ_1-axis. The resulting triangles will be called regular. Perform an additional triangulation of the quadrilaterals cut off from the regular triangles by the broken line $\Gamma(\Omega)$. All possible cases of intersection of $\Gamma(\Omega)$ and the regular triangles are shown in Fig. 1; the additional triangulation is also indicated there by dashed lines.

We shall call the following points nodes: a) mesh nodes belonging to $\overline{\Omega}$; b) vertices of the polygon Ω; c) points of intersection of a side of $\Gamma(\Omega)$ with mesh lines and with the diagonals of squares, if these intersections consist of a single point.

We shall call complexes the minimal sets of nodes lying on $\Gamma(\Omega)$ such that, together with each node, the set contains a node at a distance from it less than Gh^2.

For sufficiently small h, the constructed geometric objects have the following properties:

1) The lengths of the sides of $\Gamma(\Omega)$ lie within the limits

$$(P-1)h \leq l_i \leq (2P+3)h.$$

2) At each mesh node in $\overline{\Omega}$ there is a regular triangle belonging to $\overline{\Omega}$ with a vertex at this node.

3) Suppose that, by means of the normals to $\Gamma(\omega)$, a one-to-one correspondence has been established between the points of $\Gamma(\omega)$ and $\Gamma(\Omega)$. Let $s(\omega)$ and $s(\Omega)$ be the arc lengths of the contours $\Gamma(\omega)$ and $\Gamma(\Omega)$, respectively. Then

$$C^0 \leq ds(\Omega)/ds(\omega) \leq C^0.$$

4) All constructed triangles have the property that the ratio of the smallest side to the height is bounded by a constant R.

5) A complex cannot consist of more than three nodes, and no node can belong simultaneously to two complexes.

Introduce the class Ξ of functions $\tilde{u}(\xi_1, \xi_2)$ that are linear in the triangles and continuous in Ω, specifying the values of $\tilde{u}$ at the nodes as parameters. In doing so, we shall observe the following rules: a) the values of $\tilde{u}$ must be identical at all nodes of a complex; b) at the nodes of a complex lying at a distance less than Gh^2 from some interior node, $\tilde{u}$ must be assigned the same value as at this interior node.

We obtain the system of difference equations by seeking, in the class Ξ, the minimum of the functional *

* If the coefficients a_{ij}, c, σ have two bounded derivatives, then in (3) they may be replaced by the corresponding piecewise-linear functions in the triangles of the triangulation. In this case the order of accuracy of the scheme will not deteriorate, and the necessary computation of the integrals will become elementary.

$$\Phi_\Omega(u) + 2\iint_\Omega uf\, d\xi_1\, d\xi_2. \tag{3}$$

Here f has been extended by zero outside ω,

$$\Phi_\Omega(u) = \iint_\Omega \left(a_{ij} u_{\xi_i} u_{\xi_j} + cu^2\right) d\xi_1\, d\xi_2 + \int_{\Gamma(\Omega)} \sigma_\Omega u^2\, ds(\Omega),$$

$$\sigma_\Omega = \sigma\, ds(\omega)/ds(\Omega).$$

II. Let us estimate the largest ($\Lambda_{\max}$) and the smallest ($\Lambda_{\min}$) eigenvalues, with respect to the parameters of the function $\tilde{u}$, of the quadratic form $\Phi_\Omega(\tilde{u})$. We note the following facts:

a) The following inequality holds (with constants independent of h):

$$\Phi_\Omega(u) \geq C_3 \|u\|^2_{1,\Omega} \geq C_4 \left[\iint_\Omega u^2\, d\xi_1\, d\xi_2 + \int_{\Gamma(\Omega)} u^2\, ds(\Omega)\right].$$

b) With each grid node belonging to $\overline{\Omega}$ there is associated a regular triangle Δ. Inside it $\tilde{u}$ is linear; therefore the estimate

$$\iint_\Delta \tilde{u}^2\, d\xi_1\, d\xi_2 \geq \frac{h^2}{24}\left(\tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2\right)$$

holds, where $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are the values of $\tilde{u}$ at the vertices of Δ.

c) The length of the minimal segment ab between nodes of two neighboring complexes is greater than Gh^2. Consequently,

$$\int_{ab} \tilde{u}^2\, ds(\Omega) \geq C_5 h^2 \left(\tilde{u}_a^2 + \tilde{u}_b^2\right),$$

where $\tilde{u}_a$ and $\tilde{u}_b$ are the values of $\tilde{u}$, respectively, at the nodes of one and of the other complex. From a), b), c) follows the inequality

$$\Phi_\Omega(\tilde{u}) \geq C_6 h^2 \sum \tilde{u}_k^2,$$

where the summation is over all distinct values $\tilde{u}_k$, the parameters of the function $\tilde{u}$. The latter inequality gives the estimate $\Lambda_{\min} \geq C_6 h^2$.

Let, in the triangle $\widetilde{\Delta}$ with vertices a, b, c, the largest side be ab, the smallest bc; let l be the altitude dropped from the vertex c. Then, owing to the linearity of $\tilde{u}$,

$$\iint_{\widetilde{\Delta}} (\nabla\tilde{u})^2\, d\xi_1\, d\xi_2 = |ab|\,|l| \left\{\left(\frac{\tilde{u}_b - \tilde{u}_a}{|ab|}\right)^2 + \left[\frac{(\tilde{u}_b - \tilde{u}_a)|bc|\cos(ab, \widehat{bc})}{|ab|\,|l|} + \left(\frac{\tilde{u}_c - \tilde{u}_b}{|l|}\right)\right]^2\right\}. \tag{4}$$

We note that $|ab| \le \sqrt{2}\, h$, and $|bc| \le R|l|$ by property 4). In addition, by the construction of the class Ξ, $\tilde{u}_c - \tilde{u}_b = 0$ if $R|l| < Gh^2$. Then from (4) we obtain

$$\iint_{\tilde{\Delta}} (\nabla \tilde{u})^2 \, d\xi_1 \, d\xi_2 \le C_7 h^{-1} \left(\tilde{u}_a^2 + \tilde{u}_b^2 + \tilde{u}_c^2 \right).$$

From this estimate and from the obvious inequalities

$$\Phi_\Omega(\tilde{u}) \le C_8 \left[\iint_\Omega (\nabla \tilde{u})^2 \, d\xi_1 \, d\xi_2 + \int_{\Gamma(\Omega)} \tilde{u}^2 \, ds(\Omega) \right],$$

$$\int_{\Gamma(\Omega)} \tilde{u}^2 \, ds(\Omega) \le C_9 \sum \tilde{u}_k^2$$

there follows the estimate

$$\Phi_\Omega(\tilde{u}) \le C_{10} h^{-1} \sum \tilde{u}_k^2,$$

indicating that $\Lambda_{\max} = O(h^{-1})$.

Thus, the following theorem is valid.

Theorem 1. *The condition number of the matrix of the difference system is characterized by the estimate* $\Lambda_{\max}/\Lambda_{\min} = O(h^{-3})$.

III. The accuracy estimate for the difference scheme is carried out as follows:

1. It is shown that, for the solution $\hat{u}$ of problem (1)–(2) and the function u realizing the minimum of the functional (3) in $W_2^2(\hat{\Omega})$, the inequality holds

$$\Phi_{\hat{\Omega}}(u - \hat{u}) \le C_{11} \|u - \hat{u}\|_{1,\Omega}^2 \le C_{12} h^2 \|\hat{u}\|_{2,\omega}^2. \tag{5}$$

2. The quantity $\Phi_{\hat{\Omega}}(u - \tilde{u})$ is estimated, where $\tilde{u}$ minimizes (3) in the class Ξ.

Let $\tilde{v}$ be the function of the class Ξ constructed from the values of $\hat{u}$ at the nodes. Then, obviously,

$$\Phi_{\hat{\Omega}}(u - \tilde{u}) \le \Phi_{\hat{\Omega}}(u - \tilde{v}) \le 2\Phi_{\hat{\Omega}}(u - \hat{u}) + 2\Phi_{\hat{\Omega}}(\hat{u} - \tilde{v}).$$

It can be shown that

$$\Phi_{\hat{\Omega}}(\hat{u} - \tilde{v}) \le C_{13} h^2 \|\hat{u}\|_{2,\omega}^2. \tag{6}$$

The derivation of the last inequality is the central point of the proof.

3. It is easy to see that

$$\|\hat{u}-\tilde{u}\|_{1,\omega}^2 \leq C_{14}\Phi_{\hat{\Omega}}(\hat{u}-\tilde{u}) \leq 2C_{14}\left[\Phi_{\hat{\Omega}}(\hat{u}-u)+\Phi_{\hat{\Omega}}(u-\tilde{u})\right].$$

Estimates (5), (6) lead to the following theorem.

Theorem 2. *The accuracy of the difference scheme is characterized by the estimate*

$$\|\hat{u}-\tilde{u}\|_{1,\omega} \leq C_{15}h\|f\|_{0,\omega}.$$

The accuracy estimate cannot be improved in order. It is evident that, by a suitable numbering of the nodes, one can ensure that the matrix of the difference system is block tridiagonal.

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Note: Figure translations are in progress. See original paper for figures.