ON SOME PROPERTIES OF \(\{k; n\}_q\)-ARCS IN GALOIS PLANES

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Abstract

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ON SOME PROPERTIES OF $\{k;n\}_q$-ARCS IN GALOIS PLANES

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1. Let a projective plane $S_{2,q}$ over the Galois field $GF(q) = \gamma_q$ of order $q = p^h$ be given, where p is a prime number (the characteristic of the field) and h is a natural number. The Galois plane $S_{2,q}$ contains $q^2 + q + 1 = Q$ points and the same number of lines; each line contains $q + 1$ points, and through each point of the plane pass $q + 1$ lines.

An arbitrary set of k points of the plane $S_{2,q}$ will be called a k-system in $S_{2,q}$, $k \leq Q$. To each k-system K in $S_{2,q}$ one may assign an integer n, where n denotes the largest number of points of K lying on one line. For k-systems in $S_{2,q}$, an important but little-developed problem is the determination of the greatest values of k as a function of various values of n and q.

For $n = 2$, k-systems are customarily called k-arcs, and for them it has been proved that $k_{\max} = q + 1$ for odd q and $k_{\max} = q + 2$ for even q (1).

If $3 \leq n \leq q$, then a k-system in $S_{2,q}$ is called a $\{k;n\}_q$-arc (the cases $n = 1$ and $n = q + 1$ at the maximal value of k are discarded as trivial) (2).

By virtue of this definition of a $\{k;n\}_q$-arc, all points of the plane $S_{2,q}$ are divided into two classes: the k points of the arc (absolute points) and the $Q - k$ points external to the arc. A line of the plane is called an l-secant of the $\{k;n\}_q$-arc if it has l common points with the arc ($l = 0, 1, \dots, n$). By the principle of duality, valid in $S_{2,q}$, dual $\{k;n\}_q^*$-arcs are also considered.

We shall next consider various Diophantine systems connected with $\{k;n\}_q$-arcs. For an arbitrary absolute point A, through which pass a_l l-secants ($l = 1, 2, \dots, n$), we have the following system of Diophantine equations (the system of absolute points):

$$\sum_{l=1}^{n} a_l = q + 1, \qquad \sum_{l=2}^{n} (l-1) a_l = k - 1. \tag{1}$$

To each solution $(\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni})$ of system (1) in positive integers there corresponds a type of absolute point $A_{\alpha_{1i} \cdots \alpha_{ni}}$ and a type of homogeneous arc

$K_{\alpha_{1i}\cdots\alpha_{ni}}$ $(i = 1, 2, \dots, m)$. A nonhomogeneous $\{k; n\}_q$-arc consists of points of different types.

Similarly, we have the system of external points of the arc:

$$\sum_{l=0}^{n} b_l = q + 1, \qquad \sum_{l=1}^{n} l b_l = k, \tag{2}$$

where b_l is the number of l-secants passing through an external point. To each solution $(\beta_{0j}, \beta_{1j}, \dots, \beta_{nj})$ of system (2) in positive integers there corresponds a type of external point $B_{\beta_{0j}\cdots\beta_{nj}}$ $(j = 1, 2, \dots, m')$.

If a $\{k; n\}_q$-arc contains k_i absolute points of the i-th type, then the number of l-secants of the arc is determined by the formula

$$N_l = \sum_{i=1}^{m} k_i \alpha_{li}/l \quad (l = 1, 2, \dots, n); \qquad N_0 = Q - \sum_{l=1}^{n} N_l. \tag{3}$$

If an arbitrary l'-secant of a $\{k; n\}_q$-arc contains x_j external points and y_i absolute points, then for it there is the following l'-secant system of Diophantine equations with m unknowns:

$$\sum_{j=1}^{m'} x_j = q - l' + 1,$$

$$\sum_{j=1}^{m'} (\beta_{e'j} - 1) x_j = N_{l'} - \sum_{i=1}^{m} y_i (\alpha_{l'i} - 1) - 1, \tag{4}$$

$$\sum_{j=1}^{m'} \beta_{lj} x_j = N_l - \sum_{i=1}^{m} y_i \alpha_{li}, \qquad (l = 0, 1, 2, \dots, n; \quad l \neq l').$$

2. For $\{k; n\}_q$-arcs in $S_{2,q}$ $(n > 2)$, Barlotti proved the following $(^2)$:

$$\text{if } q \equiv 0 \pmod{n}, \quad \text{then } k \leq (n-1)q + n; \tag{I}$$

$$\text{if } q \not\equiv 0 \pmod{n}, \quad \text{then } k \leq (n-1)q + n - 2. \tag{II}$$

We also have another proof of Barlotti' s formulas, based on the Diophantine systems associated with $\{k; n\}_q$-arcs. For certain particular values of n and q, the greatest value for the number k does not attain the indicated bound (I) or (II). For example, it is known that for $n = q$ the bound (I) is attained $(^2)$, while for $n = 3,\ q = 9$ it is not attained $(^3)$; the bound (II) is attained for $n = 3$ $(^4)$.

In the present note the case $n = q/2$ $(p = 2)$ is indicated, in which the bound (I) is actually attained, and it is proved that for $n = 4$ $(p \neq 2)$ and $n = q-1$ $(q \geq 5)$ the bound (II) is not attained; moreover, for $n = 4$ $(p \neq 2)$ an upper bound for k is found, which is attained in some cases; the corresponding examples of $\{k; n\}_q$-arcs in $S_{2,q}$ are given.

In the plane $S_{2,q}$ $(q = 2^h,\ h > 2)$, *in the case* $n = q/2$, *the bound (I) is attained, i.e. there exist* $\{(q^2 - q)/2;\ q/2\}_q$*-arcs.*

Indeed, in a finite plane of characteristic 2 there exists a $(q + 2)$-arc, which contains $(q^2 + 3q + 2)/2$ points of the plane $S_{2,2^h}$. Using the Diophantine system of external lines $(l' = 0)$ for a $(q+2)$-arc, which follows from (2) by the principle of duality in the plane $S_{2,q}$, it is not difficult to show that the set $Q - (q^2 + 3q + 2)/2$ of points forms a $\{(q^2 - q)/2;\ q/2\}_q$-arc in $S_{2,2^h}$. Consequently, the bound (I) is indeed attained for $n = q/2,\ q = 2^h$ $(h > 2)$. The construction of an example of such an arc follows directly from the very method of proving its existence.

Case $n = 4$ $(p \neq 2)$**.** From the system of absolute points (1) for a $\{3q + 2; 4\}_q$-arc there follows the possibility of the existence of two types of points $A_{002,q-1}$ and A_{010q}. The nonexistence of homogeneous arcs follows directly from formulas (3).

For an inhomogeneous arc we distinguish two cases: $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$.

Consider the case $q \equiv 1 \pmod 4$. In this case $(q-1)/4$ different inhomogeneous arcs are possible, and formulas (3) can be given the following form:

$$N_1 = 0; \qquad N_2 = (3q - 12t + 5)/2; \qquad N_3 = 8t - 2;$$

$$N_4 = (3q^2 + 2q - 12t + 3)/4; \qquad N_0 = (q^2 - 4q + 4t - 1)/4 \tag{5}$$

$$(t = 1, 2, \dots, (q-1)/4).$$

From the system of external points (2) for a $\{3q + 2; 4\}_q$-arc we obtain a table of the types of external points lying on external lines. Using this table and equalities (5), we compose the system of an external line ((4) for

$l' = 0$), from which for consideration we take the following two equations:

$$\sum_{i=1}^{\frac{q^2+2q-3}{16}} x_i = q + 1,$$

$$\sum_{i=\frac{q+1}{2}}^{q-3} x_i + 2\sum_{i=q-2}^{\frac{3q-15}{2}} x_i + \dots + \frac{q-9}{4}\sum_{i=\frac{q^2+2q-83}{16}}^{\frac{q^2+2q-35}{16}} x_i + \frac{q-5}{4}\sum_{i=\frac{q^2+2q-19}{16}}^{\frac{q^2+2q-3}{16}} x_i = \frac{q^2-4q+4t-5}{4}.$$

These equations are inconsistent for $q \neq 5$ and for any $t = 1, 2, \dots, (q-1)/4$; by another method the nonexistence of the arcs under consideration is also proved for $q = 5$.

The proof of the nonexistence of $\{3q+2; 4\}_q$-arcs for the case $q \equiv 3 \pmod 4$ is carried out by an analogous method. Consequently, we have proved that $k \leq 3q+1$ for all odd $q \geq 5$.

Let us give an example of a nonhomogeneous arc with the found bound for $q = 7$ (it has been proved that no homogeneous arcs with this bound exist). Investigation of the structure of the external lines for the arc under consideration by means of the Diophantine systems (2) and (4) shows that all external points of the arc lie on 7 external lines forming a dual 7^*-arc in $S_{2,7}$. Consequently, a nonhomogeneous $\{22; 4\}_7$-arc is the set of those points of the plane which supplement the points of the 7^*-arc to the set of all points of the plane $S_{2,7}$.

Case $n = q-1$. Here the following theorem has been proved:

For all $q \geq 5$ in the plane $S_{2,q}$, $\{k; q-1\}_q$-arcs can exist only for $k \leq q^2 - q - 4$.

For $q = 5$ this bound is actually attained. An example of a homogeneous $\{16; 4\}_5$-arc, consisting of points of type A_{0033}, is constructed as follows: from the plane $S_{2,5}$ we remove a dual 3^*-arc together with all its points; it is not hard to see that the set of the remaining points of the plane $S_{2,5}$ forms the required arc.

3. Completeness of $\{k; n\}_q$-arcs. A $\{k; n\}_q$-arc in the plane $S_{2,q}$ is called **complete** if it is not contained in any $\{k'; n\}_q$-arc for any $k' > k$.

Obviously, any $\{k; n\}_q$-arc with maximal value of k is complete; therefore we shall always assume that $k < k_{\max}$.

For $n = 2$ it is known that, for $q = 4d+3$ $(d \neq 0)$, complete $(q+5)/2$-arcs ([5]) exist, and for $q = 2^{2h+1}$ $(h \geq 2)$ complete $(q+4)/2$-arcs ([6]) exist.

Here we shall prove the following theorem:

If $q = p^{2h}$, then in $S_{2,q}$ there exist complete $\{p^{2h} + p^h; p^h\}_q$-arcs.

Proof. It is not hard to see that the plane $S_{2,p^{2h}}$, constructed over the field $GF(p^{2h})$, contains the plane S_{2,p^h}, constructed over the field $GF(p^h)$, as its subplane. Further, the $p^{2h} + p^h + 1$ lines of the subplane S_{2,p^h}, considered simultaneously as lines of the plane $S_{2,p^{2h}}$, contain all points of the plane $S_{2,p^{2h}}$. Indeed, each line of the subplane S_{2,p^h} contains $p^h + 1$ points of this subplane and $p^h(p^h - 1)$ points of the plane $S_{2,p^{2h}}$ not contained in S_{2,p^h}. The total number of the latter points on the $p^{2h} + p^h + 1$ lines under consideration is equal to $p^h(p^{3h} - 1)$, which exactly corresponds to the num-

of all points of $S_{2,p^{2h}}$ not contained in S_{2,p^h}. Consequently, the plane $S_{2,p^{2h}}$, outside the lines forming the subplane S_{2,p^h}, has no points.

We shall consider the set of points of the subplane S_{2,p^h} without one of its lines l_∞, which we denote by A_{2,p^h} (the finite affine plane); it contains p^{2h} points. On the line l_∞ take any p^h points not belonging to the subplane S_{2,p^h}, which is always possible ($p^{2h} - p^h \geq p^h$). Obviously, the set of points consisting of the p^{2h} points of A_{2,p^h} and the chosen p^h points of the line l_∞ forms a $\{p^{2h} + p^h;\, p^h\}_q$-arc in $S_{2,p^{2h}}$. From the very method of constructing the arc it is clear that through any point of the plane $S_{2,p^{2h}}$ there passes at least one p^h-secant, whence the completeness of the arc follows.

I have also proved that, in the case of subplanes of the plane $S_{2,p^{xh}}$ $(x > 2)$, complete $\{p^{xh} + p^h;\, p^h\}_q$-arcs cannot exist.

In conclusion, I consider it my pleasant duty to express my heartfelt gratitude to Prof. G. B. Gurevich for his attention and interest in this work.

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Note: Figure translations are in progress. See original paper for figures.

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