ON THE UNIQUENESS OF THE SOLUTION OF AN INVERSE PROBLEM REPRESENTED BY AN INTEGRAL EQUATION OF THE FIRST KIND

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.948

MATHEMATICS

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ON THE UNIQUENESS OF THE SOLUTION OF AN INVERSE PROBLEM REPRESENTED BY AN INTEGRAL EQUATION OF THE FIRST KIND

(Presented by Academician M. A. Lavrent’ ev, July 1, 1965)

1°. In solving ill-posed problems of mathematical physics, in particular the so-called inverse problems of potential theory, one of the principal points is the proof of uniqueness of the solution (^{1,2,9}). In investigations of inverse problems of potential theory, an important role is played by problems of determining the shape of a body from its potential, if the density of the body is known, as well as the unique determination of a certain functional from the potential when the density and the domain are unknown ($^{3-6}$). In a number of cases it becomes necessary to determine the density of a substance from the potential of a given body*. The solution of this problem is equivalent to the solution of a class of integral equations of the first kind, to which the works (^{1,2,9}) and others are devoted.

2°. Let A be an open set of the space E^n $(n \geq 2)$. We denote the metaharmonic potential (^{5,6}) with density $\mu(y)$, nonzero almost everywhere in A, by

$$V_\mu(x) = \int_A \mu(y) K(x,y)\, dy, \tag{1}$$

where $dy = dy_1 \ldots dy_n$ is the volume element, and $K(x,y)$ is the fundamental solution of the metaharmonic equation

$$\Delta U - \chi^2 U = 0 \qquad (\chi = \text{const} \geq 0), \tag{2}$$

with the additional requirement that, in the case $\chi > 0$, the kernel $K(x,y)$ have the corresponding decay at infinity.

In the present paper the following problem is considered:
It is required to determine the density $\mu(y)$ for points $y \in A$, if the exterior potential $V_\mu(x)$ is known for points $x \in E^n \setminus \overline{A}$.

This problem, generally speaking, does not have a unique solution. The aim of the present paper is to single out classes of densities for which the solution is unique**.

3°. Let T be a finite multiply connected domain whose boundary belongs to the class $A^{(1,\lambda)}$.

Theorem 1 (case $\chi \geq 0$). *If for functions* $\mu_\alpha \in C^1(\overline{T})$, *which do not depend on one and the same coordinate* y_k,

$$\partial\mu_\alpha/\partial y_k = 0 \qquad (\alpha = 1, 2), \tag{3}$$

external potentials coincide, i.e.,

$$V_{\mu_1}(x) = V_{\mu_2}(x) \quad \text{for } x \in E^n \setminus \overline{T}, \tag{4}$$

then

$$\mu_1(y) = \mu_2(y) \quad \text{for } y \in \overline{T}.$$

Proof. By the hypothesis of the theorem the external potentials coincide (4); therefore, for any metaharmonic function U in a domain $D \supset \overline{T}$, the equality (4-6)

$$\int_T \mu_1(y)U(y)\,dy = \int_T \mu_2(y)U(y)\,dy \tag{5}$$

holds.

Let $\mathbf{q} = \{0, \dots, q_k, \dots, 0\}$ denote a constant vector whose direction coincides with the direction of the y_k-axis. Substituting into equality (5), as U, the function

$$U = q_k \partial H/\partial y_k,$$

where $H(y)$ is a metaharmonic function in the domain D, we obtain

$$\int_S \mu_1(y)H(y)(\mathbf{R}_y, \mathbf{n}_y)\,d_{yS} = \int_S \mu_2(y)H(y)(\mathbf{R}_y, \mathbf{n}_y)\,d_{yS}, \tag{6}$$

where $(\mathbf{q}, \mathbf{n}_y)$ is the scalar product of the vector $\mathbf{q}$ with $\mathbf{n}_y$, the unit vector of the outward normal to the boundary S at the point y. Define the function $f(y)$ on the boundary S of the domain T as follows:

$$f(y) = \operatorname{sign}\{\mu(y)(\mathbf{q}, \mathbf{n}_y)\} \quad \text{for } y \in S. \tag{7}$$

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* We note that, under some restrictions on the density μ, approximate methods for solving such problems are available in works on geophysics ([7,8]).

** The problem was posed by M. M. Lavrent' ev at the joint seminar of the Siberian Branch of the Academy of Sciences of the USSR. The results of the present note were reported by the author at the seminar of the Department of Function Theory of the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR in February 1964.

Equality (6) can be extended to the function H_f, metaharmonic in the domain T and assuming on the boundary S almost everywhere the values (7). Denoting $\mu(y) = \mu_1(y) - \mu_2(y)$, we obtain from (5)

$$\int_S \mu(y) H_f(y)(\mathbf{q}, \mathbf{n}_y)\, d_{yS} = 0. \tag{8}$$

Substituting into equality (8) the boundary conditions for the function H_f from (7), we finally obtain

$$\int_S |\mu(y)|\, |(\mathbf{q}, \mathbf{n}_y)|\, d_{yS} = 0.$$

It follows from this equality that $\mu(y) = 0$ for those points $y \in S$ for which $(\mathbf{q}, \mathbf{n}_y) \neq 0$. From this assertion, by virtue of the continuity of the function $\mu(y)$ in the closed domain $\overline{T}$, the assertion of the theorem follows.

Let (ρ, θ) denote the spherical coordinates of the point x of the space E^n $(n \geq 2)$.

Theorem 2 (case $\chi = 0$). If for functions $\mu_\alpha \in C^1(\overline{T})$ the relation

$$\partial \mu_\alpha(y)/\partial \rho = 0 \quad (\alpha = 1, 2), \quad y \in \overline{T} \tag{9}$$

holds for some choice of the origin $O \in E^n \setminus \overline{T}$, and, moreover, the external potentials coincide, i.e.,

$$V_{\mu_1}(x) = V_{\mu_2}(x) \quad \text{for } x \in E^n \setminus \overline{T}, \tag{10}$$

then

$$\mu_1(y) = \mu_2(y) \quad \text{for } y \in \overline{T}.$$

Proof. From condition (10) it follows that, for any harmonic function U in a domain $D \supset \overline{T}$, the equality

$$\int_T \mu_1(y) U(y)\, dy = \int_T \mu_2(y) U(y)\, dy \tag{11}$$

holds.

Putting

$$U = \sum_{k=1}^{n} \frac{\partial}{\partial y_k}(y_{kH}),$$

where $H(y)$ is a harmonic function in the domain $D \supset \overline{T}$, and denoting $\mu(y) = \mu_1(y) - \mu_2(y)$, we obtain

$$\int_{\overline{T}} \left[\mu(y) \sum_{k=1}^{n} \frac{\partial}{\partial y_k}(y_{kH}) \right] dy = 0. \tag{12}$$

By virtue of condition (9), equality (12) is transformed into the form

$$\int_S \mu(y) H(y) (\mathbf{R}_y, \mathbf{n}_y)\, d_{yS} = 0, \tag{13}$$

where $(\mathbf{R}_y, \mathbf{n}_y)$ is the scalar product of the radius vector $\mathbf{R}_y$ with origin at the point O and the vector $\mathbf{n}_y$. Define on the boundary S the function $f(y)$ as follows:

$$f(y) = \operatorname{sign}\{\mu(y)(\mathbf{R}_y, \mathbf{n}_y)\} \quad \text{for } y \in S. \tag{14}$$

Extending equality (13) to the function H_f, harmonic in the domain T and taking on the boundary S the values (14) almost everywhere, we obtain

$$\int_S \mu(y) H_f(y) (\mathbf{R}_y, \mathbf{n}_y)\, d_{yS} = 0. \tag{15}$$

From (14) and (15) it follows that

$$\int_S |\mu(y)|\, |(\mathbf{R}_y, \mathbf{n}_y)|\, d_{yS} = 0.$$

From this equality it follows that $\mu(y) = 0$ at those points of the boundary S for which $(\mathbf{R}_y, \mathbf{n}_y) \neq 0$. By the continuity of $\mu(y)$ in the closed domain $\overline{T}$, we obtain that $\mu(y) = 0$ for $y \in \overline{T}$. The theorem is proved.

4°. Let now A be an open set consisting of a finite number of bounded domains.

Theorem 1′ (case $\chi \geq 0$). If for functions $\mu_\alpha \in C^1(\overline{A})$, independent of one and the same coordinate y_k,

$$\partial \mu_\alpha / \partial y_k = 0 \quad (\alpha = 1, 2), \tag{16}$$

the exterior potentials coincide, i.e.

$$V_{\mu_1}(x) = V_{\mu_2}(x) \quad \text{for } x \in E^n \setminus \overline{A}, \tag{17}$$

then

$$\mu_1(y) = \mu_2(y) \quad \text{for } y \in \overline{A}.$$

Proof. For the case when $\overline{A}$ is not a connected set, construct a set $A^* \subset E^n \setminus \overline{A}$ such that the set $\overline{A^*} \cap \overline{A}$ consists of a finite number of points; moreover, by construction the set A^* consists of a sum of domains whose boundaries, with the possible exception of a finite number of points, belong to the class $A^{(1,\lambda)}$, and, in addition, $\overline{A} \cup \overline{A^*}$ is a connected set. Denote $B = A \cup A^*$. Consider the potential of density (1) of the set A^*, which we denote by $V_{A^*}(x)$. From condition (17) of the theorem it follows that

$$V_{\mu_1}(x) + V_{A^*}(x) = V_{\mu_2}(x) + V_{A^*}(x) \quad \text{for } x \in E^n \setminus \overline{B}. \tag{18}$$

Introduce the functions $\tilde{\mu}_\alpha(y)$, defining them on the set $\overline{B}$ as follows:

$$\tilde{\mu}_\alpha(y) = \begin{cases} \mu_\alpha(y), & \text{for } y \in \overline{A}, \\ 1, & \text{for } y \in \overline{A}^*. \end{cases}$$

We write equality (18) in the form

$$V_{\tilde{\mu}_1}(x) = V_{\tilde{\mu}_2}(x) \quad \text{for } x \in E^n \setminus \overline{B}, \tag{19}$$

where $V_{\tilde{\mu}_\alpha}(x)$ is the potential of the set B with density $\tilde{\mu}_\alpha$. By virtue of equality (19), for any metaharmonic function H in a domain $D \supset \overline{B}$ we obtain the following equality on Γ, the boundary of the set B:

$$\int_\Gamma \tilde{\mu}_1(y) H(y)(q, n_y)\, d_{yS} = \int_\Gamma \tilde{\mu}_2(y) H_y(q, n_y)\, d_{yS}. \tag{20}$$

Next, constructing the function

$$f(y) = \operatorname{sign}\{\mu(y)(q, n_y)\} \quad \text{for } y \in \Gamma, \tag{21}$$

we extend equality (20) to the function H_f, metaharmonic on the set B and taking the values (21) almost everywhere on the boundary Γ. Repeating arguments analogous to those used in the proof of Theorem 1, we obtain that $\tilde{\mu}_1(y) = \tilde{\mu}_2(y)$ for $y \in \overline{B}$, whence the assertion of the theorem follows.

Theorem 2′ (the case $\chi = 0$). *If for the functions* $\mu_\alpha \in C^1(\overline{A})$ *the relation*

$$\partial\mu_\alpha / \partial\rho = 0 \quad (\alpha = 1, 2) \quad \text{for } y \in \overline{A}$$

holds for some choice of origin $O \in E^n \setminus \overline{A}$*, and, moreover, the exterior potentials coincide, i.e.*

$$V_{\mu_1}(x) = V_{\mu_2}(x) \quad \text{for } x \in E^n \setminus \overline{A},$$

then

$$\mu_1(y) = \mu_2(y) \quad \text{for } y \in \overline{A}.$$

The proof of Theorem $2'$ is analogous to the proof of Theorem 2; moreover, one must additionally use the construction of the set A^*, as in the proof of Theorem $1'$.

Remark. If the origin is chosen at a point $O \in \overline{A}$, then Theorem $2'$ holds when, in place of the first condition of the theorem, one assumes that the functions $\mu_\alpha(y)$ are bounded on $\overline{A}$, continuously differentiable everywhere in $\overline{A}$, except possibly at the origin, and the relation

$$\partial\mu_\alpha/\partial\rho = 0 \quad \text{for points } y \in \overline{A} \setminus 0$$

holds.

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Received
1 VII 1965

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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