ON CLOSED IMAGES OF METRIC SPACES

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Abstract

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ON CLOSED IMAGES OF METRIC SPACES

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In the present note* a necessary and sufficient condition is established for a T_1-space to be the image of a metric (complete metric) space under a closed continuous mapping (in what follows all mappings will be assumed continuous).

It follows from the Nagata–Smirnov theorem ([3,4]) that metrizable spaces are characterized among all regular spaces by the existence in them of a refining sequence of locally finite covers. The simplest examples show that neither this property nor properties close to it are preserved under closed mappings, if one speaks of systems of open sets. The point is that under closed mappings the character of the space may increase at some points.

The situation is different with systems of closed sets. The property of a space to have a refining sequence of locally finite closed covers (which also characterizes metrizability), after a slight generalization, becomes a characteristic of closed images of metric spaces.

§ 1. **Definition 1.** A system $\{F_\alpha\}$, $\alpha \in A$, of closed sets of a space X is called **hereditarily conservative** if, for any set of indices $A' \subseteq A$ and any system $\{M_\alpha\}$, $\alpha \in A'$, of closed sets of the space X, $M_\alpha \subseteq F_\alpha$, the set

$$\bigcup_{\alpha \in A'} M_\alpha$$

is closed in X.

It follows immediately from the definition that

Lemma 1. Every locally finite system of closed sets of a topological space is hereditarily conservative.

Lemma 2. The property of a system of closed sets of being hereditarily conservative is preserved under closed mappings.

The proof is carried out by a simple verification.

Lemma 3. If a sequence $\{x_n\}$ of pairwise distinct points converges in a T_1-space X, and if a system of sets $\mathfrak{A}$ is hereditarily conservative, then there exists a natural number n^* such that the star of the system $\mathfrak{A}$ with respect to the set $\{x_n\}$, $n > n^*$, consists of a finite number of sets.

Proof. If the contrary is assumed, then one can choose a subsequence of points $\{x_{n_i}\}$ and a sequence of sets $\{B_i\}$ of the system $\mathfrak{A}$ such that $x_{n_i} \in B_i$ and the sets B_i are pairwise distinct. Since the set $\{x_{n_i}\}$ is not closed, we obtain a contradiction with the hereditary conservativeness of the system $\mathfrak{A}$.

Definition 2. A sequence of closed covers $\{\mathfrak{A}_i\}$ of a space X is called **almost refining** if, for every point $x_0 \in X$, any system of sets $\{B_i\}$, $B_i \in \mathfrak{A}_i$, $x_0 \in B_i$, either

* See also the author' s preceding note ([8]).

is hereditarily conservative, or forms a net* of the space at the point x_0.

Theorem 1. In order that a T_1-space X be a closed image of a metric space, it is necessary and sufficient that the following conditions be satisfied simultaneously:

1) there exists in X an almost refining sequence of hereditarily conservative covers forming a net of the space X;
2) X is a Fréchet-Urysohn space**.

Proof. To prove the **necessity** of the first condition, suppose that S is a metric space and that the mapping $f : S \to X$ is closed. Taking in S such a sequence of locally finite closed covers $\mathfrak{A}_i$ that the diameter of each $F^i_\alpha \in \mathfrak{A}_i$ does not exceed $1/i$, we obtain, by Lemmas 1 and 2, that the image $f\mathfrak{A}_i$ of each cover $\mathfrak{A}_i$ is a hereditarily conservative cover. Relying on the fact that the mesh of the covers $\mathfrak{A}_i$ tends to zero as $i \to \infty$, it is easy to verify that the covers $f\mathfrak{A}_i$ form an almost refining sequence in X.

Sufficiency. Let $\{\mathfrak{A}_i\}$ be an almost refining sequence of hereditarily conservative covers forming a net of the space X. We consider the product

$$\prod_{i=1}^{\infty} \mathfrak{A}_i$$

of the systems $\mathfrak{A}_i$ as a set lying in the Baire space $B(\tau)$ of the corresponding weight***. The points of this set are all possible collections $\{A_i\}$ of sets A_i, one from each system $\mathfrak{A}_i$. Denote by S the set of all such collections

$$\{A_i\} \in \prod_{i=1}^{\infty} \mathfrak{A}_i,$$

each of which forms a net of some point $x_0 \in X$. Then S is a metrizable space. Assigning to a collection $\{A_i\} \in S$ that unique point $x_0 \in X$ of which it is a net, we obtain a mapping f of the space S into the space X.

1°. f is a mapping onto the whole space X. Let x_0 be an arbitrary point of the space X. If x_0 is isolated, then, since

$$\bigcup_{i=1}^{\infty} \mathfrak{A}_i$$

forms a net of the space X, in some $\mathfrak{A}_i$ there is a one-point element $x_0 \in \mathfrak{A}_i$. Any collection $\{A_i\}$ containing x_0 is mapped to x_0. If, however, $x_0 \in [X \setminus x_0]$, then in $X \setminus x_0$ there exists a sequence $\{x_n\}$ converging to x_0, since X is a Fréchet-Urysohn space. Using the conservativity of the systems $\mathfrak{A}_i$, one can choose a sequence $\{x_{n_i}\} \subseteq \{x_n\}$ and choose from each system $\mathfrak{A}_i$ a set A_i so that $x_{n_i} \in A_i$, $x_0 \in A_i$ for all i. Since $\{x_{n_i}\}$ is a nonclosed set in X, the system $\{A_i\}$ is not hereditarily conservative and, by the condition, forms a net of the point x_0.

2°. The mapping f is continuous. This follows immediately from the definition of the topology in the space S and from the fact that each collection $\xi = \{A_i\} \in S$ forms a net of the point $f\xi$ in X.

3°. The mapping f is closed. Let N be an arbitrary set in the space S, and let M be its image in X. Suppose that M is not closed in X, and prove that N is not closed in S.

There exists a point $x_0 \in [M]_X \setminus M$. Since X is a Fréchet-Urysohn space, there exists a sequence $\{x_n\}$ of points of M converging to x_0. In the preimage of each point x_n choose a point ξ_n belonging to the set N. Denoting by O_{ix_0} the complement to the sum

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* For the definition of the concept of a net, see (1).

** For the definition of the Fréchet-Urysohn property and the preservation of this property under closed mappings, see (2).

*** For the technique of constructing mappings of sets lying in Baire space, see the papers (6, 9).

of all sets of the system $\mathfrak{A}_i$ not containing x_0 (by virtue of the conservativity of the cover $\mathfrak{A}_i$, the set $O_i x_0$ is a neighborhood of the point x_0), we obtain that all x_n, from some point on, lie in $O_i x_0$. Using Lemma 3, one can find a natural number l_1 such that the set of all elements of the system $\mathfrak{A}_1$ intersecting $\{x_n\}$, $n > l_1$, is finite, and each of them contains x_0. Therefore there exists an infinite subsequence α_1 of the sequence $\{\xi_n\}$ such that every tuple $\xi \in \alpha_1$ begins with one and the same set $A_1 \in \mathfrak{A}_1$, containing x_0. Continuing this process by induction, we find for every natural i an infinite set $\alpha_i \subseteq \{\xi_n\} \subseteq S$ and a set $A_i \in \mathfrak{A}_i$ such that $\alpha_i \subseteq \alpha_{i+1}$, every tuple $\xi \in \alpha_i$ has in the i-th place the set A_i, and each A_i contains x_0. Finally, in each α_i choose a point ξ_{n_i} so that $n_i < n_{i+1}$. By the last inequality, the sequence $\{f\xi_{n_i}\}$ converges to the point x_0. Since, moreover, for every i we have $f\xi_{n_i} \in A_i$, the system $\{A_i\}$ is not hereditarily conservative. Consequently, the tuple $\xi_0 = \{A_i\}$ forms a net of the point x_0 in the space X, i.e. $\xi_0 \in S$, $f\xi_0 = x_0$, and $\xi_0 \in N$. Obviously, the sequence $\{\xi_{n_i}\}$ converges in the space S to the point ξ_0, i.e. the set N is not closed in S. The theorem is proved.

§ 2. A countable sequence of covers $\{\mathfrak{A}_i\}$ of a topological space X is called an A-system,* if between certain elements of the covers $\mathfrak{A}_i$ a subordination relation is defined in such a way that, for all i, the following conditions are satisfied:

1. Each element $B_{i+1} \in \mathfrak{A}_{i+1}$ is subordinate to one and only one element of the cover $\mathfrak{A}_i$.
2. Each element $B_i \in \mathfrak{A}_i$ is the union of all elements of the cover $\mathfrak{A}_{i+1}$ subordinate to it.

A sequence $\{B_i\}$ of sets of an A-system $\{\mathfrak{A}_i\}$ is called a thread if, for every i, B_{i+1} is subordinate to B_i.

Theorem 2. *In order that a T_1-space X be a closed image of a metrizable space with a complete metric, it is necessary and sufficient that the following two conditions be fulfilled simultaneously:*

1) *there exists in X an A-system consisting of hereditarily conservative covers, each thread of which forms a net of some point $x \in X$;*

2) *X is a Fréchet–Urysohn space.*

The methods of proof are the same as in Theorem 1; here the metric space Ξ, whose image is our space X, may be assumed to lie in Baire space as a closed subspace (cf. in this connection (7)).

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* A-systems of covers were defined in (5).

Note: Figure translations are in progress. See original paper for figures.