A THEOREM ON HOMEOMORPHISMS AND QUASILINEAR EQUATIONS

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Abstract

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A THEOREM ON HOMEOMORPHISMS AND QUASILINEAR EQUATIONS

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In recent years M. I. Vishik (1), and then F. Browder (2), Yu. A. Dubinskii (3), and others have obtained theorems on the solvability of the first boundary-value problem for a broad class of quasilinear elliptic systems of equations of order $2m$. In these works the existence is established of generalized solutions having derivatives of order m (belonging to the space $\dot{W}_p^m(\Omega)$). A number of authors (see, for example, (4)) have expressed the wish to obtain theorems on increasing the smoothness of the corresponding solutions; however, as far as we know, such theorems have not yet been proved.

In the present paper we consider quasilinear equations of another structure, with a linear principal part, for which existence theorems are proved for solutions of increased smoothness (belonging to the space $\dot{W}_p^{m+k}(\Omega)$). Here we make essential use of theorems on a set of homeomorphisms realized by a linear elliptic operator, as well as the assumed strong ellipticity of this operator. Our considerations suggest that solutions of prescribed smoothness can occur only for quasilinear equations having the corresponding structure.

1. Consider the operator A generated by the uniformly elliptic differential expression of order $2m$ with real coefficients

$$\mathcal{L}u \equiv \sum_{|\alpha|,\,|h|=m} (-1)^{|\alpha|} D^{\alpha}\left(a_{\alpha,h}(x) D^{h} u\right) \tag{1}$$

on the space of functions belonging to $W_p^{2m}(\Omega)$ in a bounded domain $\Omega \subset R_n$ and satisfying the first boundary condition

$$u|_{\Gamma} = \partial u/\partial n|_{\Gamma} = \dots = \partial^{m-1}u/\partial n^{m-1}|_{\Gamma} = 0 \tag{2}$$

on the boundary of the domain. The space of all functions from $W_p^{2m}(\Omega)$ satisfying (2) will, as usual, be denoted by $\dot{W}_p^{2m}(\Omega)$. In (1) the following notation is used: $\alpha = (\alpha_1 \dots \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $D^{\alpha}u$ is the partial derivative of order $|\alpha|$.

The boundary of the domain Ω and the coefficients $a_{\alpha h}(x)$ are assumed to be sufficiently smooth that for the operator A the theorem on a complete set of homeomorphisms in the scale of spaces $\dot{W}_p^l(\Omega)$ is valid (see $(^{5-7})$), as is also the estimate

$$\left\|(A+\lambda I)^{-1} f\right\|_{\dot{W}_p^{2m}} \leq C\|f\|_{L_p} \tag{3}$$

with a constant C depending on p and independent of λ for $\lambda > 0$ (see (^{8}), and also (^{9})).

From inequality (3) there follows immediately the inequality

$$\left\|(A+\lambda I)^{-1} f\right\|_{L_p} \leq \frac{C_1}{\lambda}\|f\|_{L_p}. \tag{4}$$

From (3) and (4) it follows that

$$\left\|(A+\lambda I)^{-1} f\right\|_{\dot{W}_p^s} \leq$$

$$\leq C_2\left\|(A+\lambda I)^{-1} f\right\|_{L_p}^{1-s/2m}\left\|(A+\lambda I)^{-1} f\right\|_{\dot{W}_p^{2m}}^{s/2m} \leq \frac{\alpha_s}{\lambda^{1-s/2m}}\|f\|_{L_p} \quad (0 \leq s < 2m). \tag{5}$$

Writing the same inequality for p' and passing to the dual inequality for the weak extension of the operator A (denoted in the same way), we obtain

$$\left\|(A+\lambda I)^{-1} f\right\|_{L_p} \leq \frac{\alpha_s'}{\lambda^{1-s/2m}}\|f\|_{W_p^{-s}}, \tag{6}$$

where $W_p^{-s}(\Omega)$ is the space dual to $\dot{W}_{p'}^s(\Omega)$:

$$W_p^{-s}(\Omega) = \left(\dot{W}_{p'}^s(\Omega)\right)^*.$$

Applying the complex interpolation method (see (^{10})) and using results (6), from inequalities (5), (6) we pass to the inequality

$$\left\|(A+\lambda I)^{-1} f\right\|_{\dot{W}_p^{s-k}} \leq \frac{\beta_s}{\lambda^{1-s/2m}}\|f\|_{W_p^{-k}} \quad (0 \leq k \leq s < 2m). \tag{7}$$

Let us explain that, on the basis of the theorem on homeomorphisms, the operator $(A+\lambda I)^{-1}$ is a bounded operator mapping the space W_p^{-k} onto the space $\dot{W}_p^{2m-k}$; at the same time it will, of course, be a bounded operator from

W_p^{-k} into the larger space $\dot{W}_p^{s-k}$ $(s < 2m)$. Inequality (7) shows that its norm $\|(A+\lambda I)^{-1}\|_{W_p^{-k}\to\dot{W}_p^{s-k}}$ will be a quantity of order

$$O\Big(\frac{1}{\lambda^{1-s/2m}}\Big).$$

2. Let k be a fixed number, $0 \le k \le m$. Consider the boundary-value problem

$$\mathcal{L}u+\lambda u=\sum_{|l|=k}(-1)^{|l|}D^l f_l(x,D^s u) \quad (|s|\le 2m-k-1), \tag{8}$$

$$u(x)\in\dot{W}_p^m(\Omega)\cap W_p^{2m-k}(\Omega)=\dot{W}_p^{2m-k}(\Omega),$$

where $\mathcal{L}$ is the linear operator (1), $\lambda > 0$, and the argument $D^s u$ under the sign of the nonlinear function $f_l(x,D^s u)$ denotes the set of all possible derivatives of order $|s|\le\nu$, $\nu=2m-k-1$.

Concerning the functions $f_l(x,\xi_s)$, assume that they are continuous in the aggregate of the variables x,ξ_s $(x\in\bar{\Omega},\ |\xi_s|<\infty)$ and satisfy the conditions

$$|f_l(x,\xi_s)|\le\left[a(x,\xi_{s'})+b\sum_{r=r_1+1}^{2m-k-1}\sum_{|r'|=r}|\xi_{r'}|^{p_r}\right]^{1/p}, \tag{9}$$

where b is a nonnegative constant, and $a(x,\xi_{s'})$ is a nonnegative function, summable in x, and continuous and monotonically nondecreasing in the arguments $\xi_{s'}$, whose indices satisfy the inequality $|s'|\le r_1$, $r_1=2m-k-[n/p]-2$. The numbers p_r are chosen so that the embedding of the space W_p^{2m-k-1} into $W_{p_r}^r$ holds: $p_r=np/[\,n-(2m-k-1-r)p\,]$ (if the denominator is zero, then p_r is arbitrary; $p_r>1$).

If in the first part of (8) we substitute $u\in\dot{W}_p^{2m-k-1}$, then by virtue of (9), $f_l(x,D^s u)\in L_p$, and hence $\sum_{|l|=k}(-1)^{|l|}D^l f_l(x,D^s u)\in W_p^{-k}$. By virtue of the theorem on homeomorphisms, the equation

$$\mathcal{L}v+\lambda v=\sum_{|l|=k}(-1)^{|l|}D^l f_l(x,D^s u)$$

will have a unique solution $v\in W_p^{2m-k}$. Thus the operator $Bu=v$, acting from $\dot{W}_p^{2m-k-1}$ into $\dot{W}_p^{2m-k}$, is defined. A fixed point of this operator is naturally called a generalized solu-

solution of problem (8). This definition can naturally be formulated also in terms of integral identities.

By the usual arguments it is verified that the operator B is a continuous operator and, consequently, by the embedding theorems, it is a completely continuous operator acting in $\dot{W}_p^{2m-k-1}$.

From inequalities (9) there follows the estimate

$$\|f_l(x,1)u\|_{L_p} \le$$

$$\le a(\|u\|_{\dot{W}_p^{2m-k-1}}) + b_1 \sum_{r=r_1+1}^{2m-k-1} \|u\|_{\dot{W}_p^r}^{p_r/p} \le \varphi(\|u\|_{\dot{W}_p^{2m-k-1}}).$$

Then

$$\left\| \sum_{|l|=k} (-1)^{|l|} D^l f_l(x, D^s u) \right\|_{W_p^{-k}} \le C\varphi(\|u\|_{\dot{W}_p^{2m-k-1}}),$$

and, finally, by virtue of (7), for $s = 2m-1$,

$$\|Bu\|_{\dot{W}_p^{2m-k-1}} \le \left\| (A+\lambda I)^{-1} \sum_{|l|=k} (-1)^{|l|} D^l f_l(x, D^s u) \right\|_{\dot{W}_p^{2m-k-1}} \le$$

$$\le \frac{\beta_{2m-1}}{\lambda^{1/2m}} \left\| \sum_{|l|=k} (-1)^{|l|} D^l f_l(x, D^s u) \right\|_{W_p^{-k}} \le \frac{C_1}{\lambda^{1/2m}} \varphi(\|u\|_{\dot{W}_p^{2m-k-1}}). \qquad (10)$$

Choose numbers $R > 0$ and λ so large that $C_1/\lambda^{1/2m}\varphi(R) < R$. Then, by virtue of (10), the operator B maps the ball of radius R in the space $\dot{W}_p^{2m-k-1}$ into itself and, consequently, by Schauder' s principle, has a fixed point in it.

We have proved the following theorem:

Theorem 1. *For sufficiently large λ, equation (8) has at least one generalized solution in $\dot{W}_p^{2m-k}(\Omega)$.*

Theorem 1 may be regarded as a kind of theorem on increase of smoothness. The smoother the nonlinearity is, the smaller the k for which it can be represented in the form (8), and, correspondingly, the greater is the smoothness of the solution of equation (8).

3. We now consider the more general equation

$$\mathcal{L}u = \sum_{|l|=k}(-1)^{|l|}D^l f_l(x, D^s u) + \sum_{|l|=k-1}(-1)^{|l|}D^l g_l(x, D^\sigma u) \equiv F(u) \tag{11}$$

$$(|s| \le 2m-k-1, \quad |\sigma| \le 2m-k, \quad 1 \le k \le m).$$

Suppose that the functions $f_l(x,\xi_s)$ and $g_l(x,\xi_\sigma)$ satisfy the conditions

$$|f_l(x,\xi_s)| \le \left[a_1(x) + b_1\left(a_1'(x,\xi_{s'}) + \sum_{r=r_1+1}^{2m-k-1}\sum_{|r'|=r}|\xi_{r'}|^{p_r}\right)\right]^{1/p}, \tag{12}$$

$$|g_l(x,\xi_\sigma)| \le \left[a_2(x) + b_2\left(a_2'(x,\xi_{s'}) + \sum_{r=r_1+1}^{2m-k}\sum_{|r'|=r}|\xi_{r'}|^{p_r}\right)\right]^{1/p}, \tag{13}$$

where $a_1(x)$, $a_2(x)$, $a_1'(x,\xi_s)$, $a_2'(x,\xi_s)$ are nonnegative, summable with respect to x, and continuous monotonically increasing functions of the arguments $\xi_{s'}$, whose indices satisfy the inequality $|s'| \le r_1$; $r_1 = 2m-k-[n/p]-1$; b_1, b_2 are nonnegative constants.

If in the first part of (11) one substitutes a function $u \in \dot{W}_p^{2m-k}(\Omega)$ (for functions of lower smoothness $g_l(x,D^\sigma u)$ are not defined), then the second sum will be a function from W_p^{-k+1}, and the first from $W_p^{-k} \supset W_p^{-k+1}$.

Therefore, by the theorem on homeomorphisms, the solution of the equation $\mathcal{L}v = F(u)$ will belong to $\dot{W}_p^{2m-k}$. Thus, the operator $Bu = v$ will act from $\dot{W}_p^{2m-k}$ into $\dot{W}_p^{2m-k}$, but, generally speaking, will not be completely continuous. In connection with this, additional conditions are imposed on the functions $f_l(x,\xi_s)$ and $g_l(x,\xi_\sigma)$, ensuring that the conditions of the contraction mapping principle are satisfied.

Arguing analogously to the proof of Theorem 1, one can show that, for sufficiently small constants b_1 and b_2, there exists a ball $T:\ \|u\|_{\mathring{W}_p^{2m-k}} \le R$, which is mapped into itself by the operator B.

Suppose that $f_l(x,\xi_s)$ and $g_l(x,\xi_\sigma)$ are such that in the ball T

$$\|f_l(x,D^s u_1) - f_l(x,D^s u_2)\|_{L_p} \le C_1 \|u_1 - u_2\|_{\mathring{W}_p^{2m-k}}, \tag{14}$$

$$\|g_l(x,D^\sigma u_1) - g_l(x,D^\sigma u_2)\|_{L_p} \le C_2 \|u_1 - u_2\|_{\mathring{W}_p^{2m-k}}, \tag{15}$$

where C_1 and C_2 are nonnegative constants. Under these assumptions the following is true.

Theorem 2. *If the functions $f_l(x,\xi_s)$, $g_l(x,\xi_\sigma)$ satisfy conditions (12), (13) and, in addition, in the ball T, conditions (14) and (15), then, for sufficiently small constants C_1, C_2, there exists a unique generalized solution of equation (11), belonging to the space $\overset{\circ}{W}{}_p^{2m-k}(\Omega)$.*

4. If we pass to the equation

$$\mathcal{L}u = \sum_{|l|=k} (-1)^{|l|} D^l f_l(x, D^s u) + \sum_{|l|=k-2} (-1)^{|l|} D^l g_l(x, D^\sigma u)$$

$$(|s| \leq 2m-k-1,\ |\sigma| \leq 2m-k+1,\ 2 \leq k \leq m)\,,$$

then it is not difficult to see that it may have no generalized solution, since the functions $g_l(x, D^\sigma u)$ are defined only for $u \in W_p^{2m-k+1}$. In this case $\mathcal{L}u \in W_p^{-k+1}(\Omega)$, the first sum on the right-hand side belongs to $W_p^{-k}(\Omega)$, and the second to $W_p^{-k+2}(\Omega)$.

If the first sum does not belong to $W_p^{-k+1}(\Omega)$, then equality (11) is not satisfied.

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