Soviet-era science, translated into English

GENERALIZED NORMAL CORRELATION

MATHEMATICS

1966

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196601.65333

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 519.21

MATHEMATICS

O. V. SARMANOV

GENERALIZED NORMAL CORRELATION AND TWO-DIMENSIONAL FRÉCHET CLASSES

(Presented by Academician S. N. Bernstein on 17 VIII 1965)

1. Consider the class of functions $N\{f(x,y)\}$ representable by bilinear expansions in Hermite polynomials

$$f(x,y) = \frac{\exp\left[-\frac{x^2+y^2}{2}\right]}{2\pi}\left[1+\sum_{k=1}^{\infty} c_k H_k(x) H_k(y)\right]. \tag{1}$$

The series (1) is assumed to converge in the mean, which is equivalent to the condition

$$\sum_{k=1}^{\infty} c_k^2 < \infty, \tag{2}$$

where $H_k(x)$ are orthogonal and normal with weight $e^{-x^2/2}/\sqrt{2\pi}$ on the whole real axis, i.e.

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} e^{x^2/2} \frac{d^k}{dx^k}\left(e^{-x^2/2}\right) = \frac{1}{\sqrt{k!}}\left[x^k - \frac{k(k-1)}{2} x^{k-2} + \cdots\right]. \tag{3}$$

It follows from (1) that

$$\int_{-\infty}^{\infty} f(x,y)\,dx = \frac{e^{-y^2/2}}{\sqrt{2\pi}}; \qquad \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \tag{4}$$

Definition. When the sum of the series (1) is nonnegative for all real x and y, we shall say that there is a generalized normal correlation between x and y.

2. Theorem. *In order that the sum of the series (1) be the density of a generalized normal correlation, it is necessary and sufficient that the coefficients* c_k

SovietRxiv

be the moments of some probability distribution concentrated inside the interval $[-1, 1]$.

Necessity. Let the sum of the series (1) be nonnegative for all x and y. Find the conditional mathematical expectation of the quantity $(y/x)^k$ for fixed $x \neq 0$. Express y^k through the polynomials (3):

$$y^k = \sqrt{k!}H_k(y) + a_{k-2}H_{k-2}(y) + a_{k-4}H_{k-4}(y) + \cdots,$$

where $a_{k-2}, a_{k-4}, ...$ are fully determined constants.

$$M_x y^k = \sqrt{k!}c_k H_k(x) + a_{k-2}c_{k-2}H_{k-2}(x) + \cdots$$

$$= c_k x^k + b_{k-2}x^{k-2} + b_{k-4}x^{k-4} + \cdots,$$

where $b_{k-2}, b_{k-4}, ...$ are also certain constants. Thus,

$$M_x(y/x)^k = c_k + b_{k-2}/x^2 + b_{k-4}/x^4 + \cdots. \tag{5}$$

and for sufficiently large x, c_k differs arbitrarily little from the k-th moment of the variable y/x; moreover, by virtue of (2), c_k cannot be a moment of a variable that, with positive probability, goes outside the interval $[-1, 1]$ or assumes the values ± 1.

Sufficiency. Let $\{c_k\}$ form the sequence of moments of some variable ξ, concentrated inside the interval $[-1, 1]$ and given by the distribution function $G(\xi)$; then for each fixed $|\xi| < 1$ consider the well-known expansion for the density of normal correlation with correlation coefficient ξ

$$\frac{\exp\left[-\frac{x^2+y^2-2\xi xy}{2(1-\xi^2)}\right]}{2\pi\sqrt{1-\xi^2}} = \frac{\exp\left[-\frac{x^2+y^2}{2}\right]}{2\pi}\left[1 + \sum_{k=1}^{\infty}\xi^k H_k(x)H_k(y)\right], \tag{6}$$

where, by assumption,

$$\int_{-1}^{1}\xi^k\,dG(\xi) = c_k, \qquad k = 1, 2, ... \tag{7}$$

From (6) and (7) it follows that

$$\frac{\exp\left[-\frac{x^2+y^2}{2}\right]}{2\pi}\left[1 + \sum_{k=1}^{\infty}c_k H_k(x)H_k(y)\right] =$$

SovietRxiv

$$= \frac{1}{2\pi} \int_{-1}^{1} \frac{\exp\left[-\frac{x^2+y^2-2\xi xy}{2(1-\xi^2)}\right]}{\sqrt{1-\xi^2}} \, dG(\xi) > 0, \tag{8}$$

which completes the proof.

Remark. In normal correlation, $\xi = R$ with probability 1, and $c_k = R^k$. If ξ assumes the two values $\pm R$ with probabilities $1/2$, so that $c_{2k} = R^{2k}$, $c_{2k-1} = 0$, $k = 1, 2, ...$, the corresponding density was considered in ([1]) and was given the name pseudonormal correlation; finally, if all $c_k = 0$, then x and y are independent.

Corollary 1. If the sum (1) has constant sign and the variables x and y are dependent, then $c_{2k} > 0$ for all $k = 1, 2, ...$

Corollary 2. It is easy to verify that the characteristic function of the density (8) has the form

$$\varphi(t_1, t_2) = \exp\left[-\frac{t_1^2 + t_2^2}{2}\right] \int_{-1}^{1} \exp[-Rt_1t_2] \, dG(R). \tag{9}$$

In particular, if it is known that the correlation between x and y is normal, but the correlation coefficient is not known exactly and is uniformly distributed on the interval $[R - \varepsilon, R + \varepsilon]$, $R - \varepsilon > -1$, $R + \varepsilon < 1$, then the characteristic function of such a generalized normal correlation is expressed with the aid of (9) as follows:

$$\varphi(t_1, t_2, \varepsilon) = \exp\left[-\frac{t_1^2 + t_2^2 + 2Rt_1t_2}{2}\right] \frac{\operatorname{sh}(\varepsilon t_1 t_2)}{\varepsilon t_1 t_2}. \tag{10}$$

2. M. Fréchet in ([2], [3]) posed the following general problem: to find the class of two-dimensional distribution functions $\{P(x, y)\}$ having given partial (extreme or marginal) one-dimensional distributions $F(x)$ and $F_1(y)$. In the preceding section an important part of the normal Fréchet class has been found.

Morgenstern and Gumbel ([4]) considered the following example of functions belonging to the Fréchet class $K\{F(x), F_1(y)\}$ for arbitrary given $F(x)$ and $F_1(y)$:

$$P(x, y) = F(x)F_1(y) \left[1 + \lambda(1 - F(x))(1 - F_1(y))\right], \qquad -1 < \lambda < 1. \tag{11}$$

If, however, densities exist (denoted by the corresponding lowercase letters), then

$$p(x, y) = f(x)f_1(y) \left[1 + \lambda(2F(x) - 1)(2F_1(y) - 1)\right]; \tag{12}$$

SovietRxiv

in particular, in the normal case,

$$p(x,y) = \frac{\exp\left[-\frac{x^2+y^2}{2}\right]}{2\pi}\,[1 + 4\lambda\Phi(x)\Phi(y)], \tag{13}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}\,dt.$$

3. Let us consider the following more general method of constructing functions from the Fréchet class $k\{f(x), f_1(y)\}$, specified by the densities $f(x)$ and $f_1(y)$ on $L = [a \le x \le b]$ and $L_1 = [a_1 \le y \le b_1]$. Let $\varphi(x)$ be an arbitrary function (not equal to a constant) bounded on L, and let $\varphi_1(y)$ be an arbitrary function bounded on L_1. Then the quantities

$$m = \int_a^b \varphi(x)f(x)\,dx, \qquad m_1 = \int_{a_1}^{b_1} \varphi_1(y)f_1(y)\,dy,$$

$$\sigma^2 = \int_a^b [\varphi(x) - m]^2 f(x)\,dx, \qquad \sigma_1^2 = \int_{a_1}^{b_1} [\varphi_1(y) - m_1]^2 f_1(y)\,dy \tag{14}$$

are meaningful.

Denote

$$h = \sup_{x\in L}\frac{|\varphi(x) - m|}{\sigma}, \qquad h_1 = \sup_{y\in L_1}\frac{|\varphi_1(y) - m_1|}{\sigma_1}. \tag{15}$$

The one-parameter family of densities

$$p_1(x,y,\lambda) = f(x)f_1(y)\left[1 + \frac{\lambda}{hh_1}\,\frac{[\varphi(x) - m][\varphi_1(y) - m_1]}{\sigma\sigma_1}\right], \tag{16}$$

where $-1 \le \lambda \le 1$, belongs to the Fréchet class $k\{f(x), f_1(y)\}$, since $\varphi(x) - m$ and $\varphi_1(y) - m_1$ are orthogonal to unity with weights $f(x)$ and $f_1(y)$, and the sum (16) is nonnegative.

We note that, by virtue of (14), $h \ge 1$ and $h_1 \ge 1$, and

$$\rho = \lambda/hh_1 \tag{17}$$

is the maximal correlation coefficient (5) between x and y.

SovietRxiv

Integrating (16), we pass to the distribution functions

$$P_1(x,y,\lambda) = F(x)F_1(y) + \frac{\lambda}{hh_1}\int_a^x \frac{\varphi(u)-m}{\sigma}\,dF(u)\int_{a_1}^y \frac{\varphi_1(v)-m_1}{\sigma_1}\,dF_1(v). \tag{18}$$

Since $F(x)$ and $F_1(y)$ are bounded functions, one may put $\varphi(x) \equiv F(x)$, and $\varphi_1(y) \equiv F_1(y)$; then, for continuous $F(x)$ and $F_1(y)$, we obtain that

$$\frac{1}{2} = \int_a^b F(x)\,dF(x) = m = m_1 = \sup_{x\in L}|F(x)-m| = \sup_{y\in L_1}|F_1(y)-m_1|,$$

$$P_1(x,y,\lambda) = F(x)F_1(y) + \lambda(F^2(x)-F(x))(F_1^2(y)-F_1(y)) =$$

$$= F(x)F_1(y)[1+\lambda(F(x)-1)(F_1(y)-1)], \tag{19}$$

as in example (11). The assumption of continuity is not a restriction, for, if it is proved that (19) is a distribution function for

for arbitrary discontinuous $F(x)$ and $F_1(y)$, then (19) gives a distribution function also for arbitrary distribution functions $F(x)$ and $F_1(y)$.

4. If the densities $p_k(x,y)$ belong to some Fréchet class, then a linear combination of them of the form $\sum_k \varepsilon_k p_k(x,y)$, where $\varepsilon_k > 0$ and $\sum_k \varepsilon_k = 1$, also belongs to the same class. With the help of this remark and (16), one can construct new examples of densities and distribution functions belonging to a given class (the possibility of similar bilinear constructions was indicated in [6]).

Let us give several examples

$$p(x,y) = f(x)f_1(y)\left[1+\sum_{k=1}^{\infty}\varepsilon_k\cos 2k\pi(F(x)-{}^1\!/_2)\cos 2k\pi(F_1(y)-{}^1\!/_2)\right], \tag{20}$$

where $\sum_{k=1}^{\infty}|\varepsilon_k| \le 1$, and, integrating (20), we obtain the corresponding family of distribution functions

$$P(x,y) = F(x)F_1(y) + \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{\varepsilon_k}{k^2}\sin 2\pi k(F(x)-{}^1\!/_2)\sin 2\pi k(F_1(y)-{}^1\!/_2). \tag{21}$$

If in (18) we put $\varphi(x) = F^k(x)$, $\varphi_1(y) = F_1^k(y)$, and then consider the linear combination of the functions obtained, putting $\varepsilon_k = \varepsilon^k$, $k = 1, 2, \ldots,\ 0 \le \varepsilon \le {}^1\!/_2$,

SovietRxiv

then we arrive at the following expression, also giving functions from the Fréchet class:

$$P_2(x,y,\varepsilon) = F(x)F_1(y)\left[1 + \frac{l(\varepsilon)}{\varepsilon}\right] - \frac{F(x)}{\varepsilon}\, l[\varepsilon F_1(y)]$$
$$-\frac{F_1(y)}{\varepsilon}\, l[\varepsilon F(x)] + \frac{1}{\varepsilon}\, l[\varepsilon F(x)F_1(y)], \tag{22}$$

where

$$l(z) = -\int_0^z \frac{\ln(1-t)}{t}\,dt = \sum_{k=1}^{\infty}\frac{z^k}{k^2}, \quad \text{and} \quad P_2(x,y,0) = F(x)F_1(y).$$

In conclusion, let us give one example of discontinuous densities from the normal Fréchet class

$$f(x,y,\lambda) = \begin{cases} \dfrac{\exp\left[-\dfrac{x^2+y^2}{2}\right]}{2\pi}(1+\lambda), & \text{for } xy > 0, \\ \dfrac{\exp\left[-\dfrac{x^2+y^2}{2}\right]}{2\pi}(1-\lambda), & \text{for } xy \le 0, \\ -1 \le \lambda \le 1. \end{cases} \tag{23}$$

For the distribution (23), λ is the maximal correlation coefficient.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
17 VIII 1965

REFERENCES

[1] O. V. Sarmanov, DAN, **132**, No. 2 (1960).
[2] M. Fréchet, Ann. Univ. Lyon, ser. III, **A14**, 53 (1951).
[3] M. Fréchet, C. R., **242**, No. 20, 2426 (1956).
[4] E. Gumbel, C. R., **246**, No. 19, 2717 (1958).
[5] O. V. Sarmanov, DAN, **121**, No. 1 (1958).
[6] S. Kantorovitz, Bull. Res. Council. Israel, **F.7**, No. 1, 43 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.