Reports of the Academy of Sciences of the USSR

1966

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1966. Vol. 169, No. 5

Mechanics

I. M. Glazman, V. N. Mitin

Detuning of Vibrational Systems as a Problem of Convex Programming

(Presented by Academician A. Yu. Ishlinskii, 27 XI 1965)

1. Let Q be a vibrational system with $m < \infty$ degrees of freedom and m stiffnesses. For definiteness of terminology, considering this system to be an elastic shafting, denote by u_k and I_k $(k = 1, 2, \dots, m)$ its stiffnesses and moments of inertia, respectively. By $(0, \beta)$ we denote the interval of squares of possible frequencies of external actions.

The problem of detuning the system Q from the resonance-dangerous zone $(0, \beta)$ consists in such a change of a prescribed number $p \leq m$ of its stiffnesses that the square λ of the smallest natural frequency of the system will satisfy the inequality $\lambda \geq \beta^*$. The set K of points $\mathbf{r}(u_1, u_2, \dots, u_p)$ of the Euclidean space E_p of stiffnesses for which $\lambda = \lambda[\mathbf{r}] \geq \beta$ will be called the **antiresonance** region. We shall call the detuning **optimal** if the stiffness vector $\mathbf{r}(u_1, u_2, \dots, u_p)$ after detuning deviates least from the vector of variable stiffnesses before detuning, which we shall denote by $\mathbf{r}^*(u_1^*, u_2^*, \dots, u_p^*)$. In other words, the problem of optimal detuning reduces to minimizing the quadratic functional $F(\mathbf{r}) = (\mathbf{r} - \mathbf{r}^*, \mathbf{r} - \mathbf{r}^*)$ under the condition $\mathbf{r} \in K$. The solution of this problem will be denoted by $\mathbf{r}_\infty(u_1^\infty, u_2^\infty, \dots, u_p^\infty)$. The problem posed arises in monitoring the dynamic strength of designed machine assemblies, as well as in cases of reconstruction of existing machine assemblies in connection with changes in their operating conditions. An analogous problem may also arise in the design of composite oscillatory LC-circuits in electrical engineering.

In the present note an algorithm $\mathfrak{R}$ is constructed for solving the stated problem of optimal detuning, and a modified algorithm $\widetilde{\mathfrak{R}}$ for solving an analogous problem with additional constraints. The constructed algorithms are implemented with the aid of rational operations alone, possess an exponential rate of convergence, and provide for termination upon achieving a prescribed degree of accuracy. Machine implementation of these algorithms makes it possible to carry out detuning of vibrational systems on an electronic digital computer.

2. Let us first consider the case $p = m$. In this case detuning is always possible, since for sufficiently large u_k $(k = 1, 2, \dots, m)$ one will have $\mathbf{r}(u_1, u_2, \dots, u_m) \in K$.

Introducing the vector $\mathbf{x}_k(x_1, x_2, \dots, x_m)$ of relative twists $x_k = x_k(t)$ of successive disks of the shafting, we obtain for the kinetic and potential energies of the system Q the expressions $T = \frac{1}{2}(A\mathbf{x}(t), \mathbf{x}(t))$ and $U = \frac{1}{2}(B\mathbf{x}(t), \mathbf{x}(t))$, where $A = (a_{jk})_1^m$ is a positive definite symmetric matrix with elements $a_{jk} > 0$, depending only on the moments of inertia I_r $(r = 1, 2, \dots, m)$, and $B = (b_{jk})_1^m$ is a diagonal matrix with elements $b_{jk} = \delta_{jk} u_k$. The Rayleigh function $R(\mathbf{x}, \mathbf{r})$ of the system Q depends on $2m$ variables—the coordinates of the amplitude vector $\mathbf{x}$ and the coordinates of the stiffness vector $\mathbf{r}$.

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* We owe this formulation of the problem to L. I. Shteinvolf.

By virtue of the variational properties of the eigenvalues for a pencil of forms, the antiresonance region K is characterized by the fact that

$$R(\mathbf{x}, \mathbf{r}) = (B[\mathbf{r}]\mathbf{x}, \mathbf{x})/(A\mathbf{x}, \mathbf{x}) \geq \beta \tag{1}$$

for all $\mathbf{x} \in E_m$. The following theorem contains a qualitative description of the region K.

Theorem 1. *The antiresonance region K lies in the first orthant $(\mathbf{r} > 0)$ of the space E_m, contains, together with each point $\mathbf{r}$, any point $\mathbf{r}' \geq \mathbf{r}$, extends to infinity, and is convex.*

The first assertion is obvious; the second follows from the monotone dependence of the natural frequencies on the stiffnesses; the third follows from the second; and, finally, convexity follows from inequality (1) and the relation

$$R(\mathbf{x}, \mu\mathbf{r}_1 + (1-\mu)\mathbf{r}_2) = \mu R(\mathbf{x}, \mathbf{r}_1) + (1-\mu) R(\mathbf{x}, \mathbf{r}_2).$$

From Theorem 1 we conclude that optimal detuning is a problem of convex programming.

3. In solving convex programming problems by known methods, at each step of the computational process one generally has to solve an auxiliary nonlinear equation. This is necessary for determining the length of the relaxation step and, in some methods, also for returning, after the next step, the point from the tangent plane to the constraint surface (see ([1],[2])). For our special problem of convex programming, it is possible to avoid solving any nonlinear equations. This is achieved by using controlling sequences from ([3]) (to determine the length of the relaxation step) and by a suitable parametrization of the constraint equations (to prevent the point from leaving the constraint surface). The method of parametrizing the boundary S of the region K is given by the following theorem.

Theorem 2. *The boundary S of the antiresonance region K is described by the parametric equations*

$$u_j x_j = \beta \sum_{k=1}^{m} a_{jk} x_k \quad (j = 1, 2, \dots, m), \tag{2}$$

where u_j are the coordinates of a point on S; $\mathbf{x}(x_1, x_2, \dots, x_m)$ is a vector parameter ranging over the orthant $\mathbf{x} > 0$.

The proof follows from Perron' s theorem, according to which, for a given $\mathbf{r}$, the positive solution $\mathbf{x}$ of the linear system of equations (2) is the eigenvector corresponding to the smallest frequency $\sqrt{\beta}$ of the system Q.

Equation (2) can be given greater geometric expressiveness if one observes that from (2) there follows the equality

$$\sum_{k=1}^{m} x_k^2 \, du_k = 0,$$

so that the coordinates of the vector parameter $\mathbf{x}$ are the arithmetic roots of the coordinates of the inward normal $\mathbf{n}(n_1, n_2, \dots, n_m)$ to S. Accordingly, equations (2) may be represented in the form $\mathbf{r} = \mathbf{r}[\mathbf{x}]$, where $x_k = \sqrt{n_k}$ $(k = 1, 2, \dots, m)$, or in the form $\mathbf{r} = \mathbf{r}[\mathbf{n}]$. In this case we may assume that

$$\|\mathbf{n}\|^2 = (\mathbf{n}, \mathbf{n}) = 1 \quad \text{or} \quad \|\mathbf{x}\|_4^4 = \sum_{k=1}^{m} x_k^4 = 1. \tag{3}$$

Passing now in $F(\mathbf{r})$ from the variable $\mathbf{r}$ to the independent variable $\mathbf{x}$ and setting $F(\mathbf{r}[\mathbf{x}]) = \Phi(\mathbf{x})$, we reduce the problem of optimal detuning to minimization of the functional

$$\Phi(\mathbf{x}) = (\mathbf{r}[\mathbf{x}] - \mathbf{r}^*,\ \mathbf{r}[\mathbf{x}] - \mathbf{r}^*) \tag{4}$$

subject to condition (3). From (2) there follows the following strengthening of Theorem 1.

Theorem 3. *The boundary S of the antiresonance region K is strictly convex.*

This theorem follows from the stronger assertion of nondegeneracy of the nonnegative quadratic form $-(d\mathbf{r}, d\mathbf{n})$, and this latter ⋯

it follows that the form

$$-\frac{1}{2}(d\mathbf{r}, d\mathbf{n}) = \sum_{j,\ k=1}^{m} (\delta_{jk} u_k - \beta a_{jk}) \, dx_{jdx} k$$

$$= \frac{\beta}{2} \sum_{j,\ k=1}^{m} {}' a_{jk} x_{jx} k \left(\frac{dx_j}{x_j} - \frac{dx_k}{x_k} \right)^2$$

obviously cannot vanish under constraint (3), unless all $dx_j = 0$.

For what follows it is essential that at the minimum point $\mathbf{r}_\infty = \mathbf{r}[\mathbf{x}_\infty]$ of the functional (4) the strict inequality

$$d^2\Phi(\mathbf{x}_\infty) > 0, \tag{5}$$

holds, as follows easily from the convexity of S.

4. To minimize the functional $\Phi(\mathbf{x})$, we shall use the method of gradient descent with subsequent normalization by formula (3). From (2) we obtain for the gradient the expression $\nabla\Phi(\mathbf{x}) = G\mathbf{r}[\mathbf{x}]$, where

$$G = \left(\frac{\beta a_{jk} - \delta_{jk} u_k}{x_j} \right)_{j,\ k=1}^{m}, \tag{6}$$

so that the working formulas of the process will have the form

$$\mathbf{x}'_{s+1} = \mathbf{x}_s - \gamma_{sG} s\mathbf{r}[\mathbf{x}_s], \tag{7}$$

$$\mathbf{x}_{s+1} = \mathbf{x}'_{s+1} / \|\mathbf{x}'_{s+1}\|_1 \quad (s = 0, 1, 2, ...). \tag{8}$$

Here in (7) we shall determine the relaxation multipliers by the algorithm described and investigated in ([3]) (with a possible additional division of the multiplier in half in the case of leaving the domain $\mathbf{x} > 0$). Then, according to (3), process (7), (8) will converge to the minimum point $\mathbf{r}_\infty = \mathbf{r}[\mathbf{x}_\infty]$, and, by virtue of condition (5), it follows from ([4,6]) that the rate of convergence will be exponential. Finally, since from the convexity of S for any $\mathbf{r} \in S$ the inequality

$$(\mathbf{r} - \mathbf{r}^*, \mathbf{n}) \leq \|\mathbf{r}_\infty - \mathbf{r}^*\| \leq \|\mathbf{r} - \mathbf{r}^*\|,$$

follows, as the stopping criterion for a prescribed degree of accuracy $\delta > 0$ one may take the inequality $\|\mathbf{r}_s - \mathbf{r}^*\| - (\mathbf{r}_s - \mathbf{r}^*, \mathbf{n}_s) < \delta$. Thus all the properties of the constructed algorithm stated in No. 1 have been established. It can also be shown that the normalization (8) is not obligatory.

5. If not all stiffnesses are varied, i.e. $p < m$, then the question arises of the existence of a solution to the detuning problem. This question is resolved with the help of the following criterion*.

Theorem 4. *In order that the vibration system Q can be detuned from the zone $(0, \beta)$ by varying p stiffnesses $u_1, u_2, \dots, u_p$, with the $m - p$ stiffnesses $u_{p+1} = u^*_{p+1}, u_{p+2} = u^*_{p+2}, \dots, u_m = u^*_m$ fixed, it is necessary and sufficient that all principal minors of the matrix $(\delta_{jk} u^*_k - \beta a_{jk})^m_{j,k=p+1}$ be positive.*

For the actual detuning one should first exclude the variables $x_{p+1}, x_{p+2}, \dots, x_m$ from (2), after which the system of equations (2) takes the form

$$u_{jx} j = \beta \sum_{k=1}^{p} \tilde{a}_{jk} x_k \quad (j = 1, 2, \dots, p). \tag{9}$$

The truncated system (9), for arbitrary $\mathbf{x}(x_1, x_2, \dots, x_p) > 0$, represents parametric equations of the boundary of the section of the antiresonance region K by the subspace $E_p \subset E_m$, if and only if the detuning-possibility criterion indicated by Theorem 4 is satisfied. Now optimal detuning can be carried out by formulas (6), (7), (8), in which one should replace m by p and a_{jk} by $\tilde{a}_{jk}$.

* Another criterion was obtained earlier in ([5]).

6. In real tuning problems it is natural to prescribe technically admissible bounds for the variation of the stiffnesses. Since, obviously, optimal tuning excludes a decrease of the initially specified stiffnesses, the restrictions on variation may be represented by the inequalities

$$u^*_j \leqslant u_j \leqslant u^*_j + b_j = u^{**}_j \quad (b_j \geqslant 0;\ j = 1, 2, \dots, m), \tag{10}$$

where nonvariable stiffnesses correspond to $b_j = 0$. Without loss of generality, we shall assume all $b_j > 0$. The problem is now reduced to minimizing $F(\mathbf{r})$ for $\mathbf{r} \in K \cap \Pi$, where Π is the parallelepiped of constraints (10). The solution of this problem, in general terms, reduces to the following.

First we determine the signs of the successive principal minors of the matrix $(\delta_{jk} u^{**}_k - \beta a_{jk})^m_{j,k+1}$. If all these minors are positive, then $K \cap \Pi \neq 0$, and the problem is solvable; otherwise $K \cap \Pi \neq 0$, and tuning is impossible. When $K \cap \Pi \neq 0$, we choose arbitrarily a path Γ consisting of one-dimensional edges of the parallelepiped Π, leading from the vertex $(u^*_1, u^*_2, \dots, u^*_m)$ to the vertex $(u^{**}_1, u^{**}_2, \dots, u^{**}_m)$, and find the point $\mathbf{r}_0$ of intersection of Γ with S (this requires only the solution of a linear equation). The subsequent process is constructed so that the successive approximations $\mathbf{r}_s = \mathbf{r}[\mathbf{x}_s]$ remain on S and do not leave Π. In this case two cases may occur.

1°. For the given s, the point $\mathbf{r}_s$ lies inside Π. Then the next step is carried out as in No. 4, but if $\mathbf{r}_{s+1}$ turns out to be outside Π, then γ_s should be decreased to the value corresponding to the point of intersection of the line $\mathbf{r}(\gamma) = \mathbf{r}[\mathbf{x} - \gamma G_s(\mathbf{r}_s - \mathbf{r}^*)]$ with the boundary of Π (this requires only the solution of linear equations).

2°. For the given s, the point $\mathbf{r}_s$ lies on the boundary of Π. Then we introduce the unit vector $\mathbf{e}_s(e_{s1}, e_{s2}, \dots, e_{sm})$ of the vector

$$G_{sG}s(\mathbf{r}[\mathbf{x}_s] - \mathbf{r}^*)$$

and verify the relation

$$\pm e_{sj} \leqslant \mu \quad (0 < \mu < 1) \tag{11}$$

for those j for which the j-th coordinate of the vector $\mathbf{r}_s$ assumes one of the extreme values in (10). The plus or minus sign is taken according to whether this extreme value is the greatest or the least, and μ is an arbitrarily small but fixed positive number. If some of the inequalities (11) are satisfied, then the stiffnesses corresponding to them are fixed at this step of the process and, varying the remaining stiffnesses, we pass to the next approximation, as described in No. 5.

To control the process, even before the beginning of the computations, a controlling sequence is prescribed; it is used as in (3), except in the case when $\mathbf{r}_{s+1}$ leaves Π and one has to proceed as indicated in 1°. It can be shown that the algorithm described above has all the properties established for the tuning algorithm without the constraints (10) in No. 4.

Physico-Technical Institute of Low Temperatures Academy of Sciences of the Ukrainian SSR Kharkov Polytechnic Institute named after V. I. Lenin

Received 25 XI 1965

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Note: Figure translations are in progress. See original paper for figures.