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Abstract

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MATHEMATICS

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ON A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH A SHIFT

(Presented by Academician I. N. Vekua on 11 IX 1965)

Let Γ be a collection of a finite number of simple closed Lyapunov contours having no common points, and let $\alpha(t)$ be a homeomorphism of the contour Γ which does not change the orientation of Γ .* We assume that there exists $\alpha'(t) \in H$, $\alpha'(t) \neq 0$ ($t \in \Gamma$); $\alpha(t)$ generates a cyclic group of order n with elements

$$\alpha_0(t) \equiv t, \quad \alpha_1(t) = \alpha(t), \quad \alpha_2(t), \dots, \alpha_{n-1}(t) \\ (\alpha_j(t) = \alpha(\alpha_{j-1}(t))).$$

Consider a singular equation of the form

$$\Lambda\varphi \equiv \sum_{\nu=0}^{n-1} \left\{ A_{\nu}^{(1)}(t_0)\varphi(\alpha_{\nu}(t_0)) + A_{\nu}^{(2)}(t_0)\overline{\varphi(\alpha_{\nu}(t_0))} + S_{1,\nu}\varphi + \overline{S_{2,\nu}\varphi} \right\} = f(t_0), \quad (1)$$

where

$$S_{i,\nu}\varphi \equiv \frac{1}{\pi i} \int_{\Gamma} \frac{K_{\nu}^{(i)}(t_0, t)\varphi(t) dt}{t - \alpha_{\nu}(t_0)}, \\ A_{\nu}^{(i)}(t_0), \quad K_{\nu}^{(i)}(t_0, t), \quad f(t_0) \in H$$

are given functions, and $\varphi(t) \in H$ is the function sought.**

Denote by $\theta(j)$ the periodic numerical function with period n (n is the order of the cyclic group generated by the mapping $\alpha(t)$), defined on the set $0, 1, 2, \dots$, which for $0 \leq j \leq n-1$ takes the value $\theta(j) = j$, and by $k(E)$ the dimension of the null space of the equation $Eg = 0$. Below, linear independence will everywhere be understood over the field of real numbers.

As in ⁽¹⁾, it is not difficult to verify that with equation (1) there is associated a system of singular integral equations

$$P\vec{\Phi} \equiv G_1(t_0)\vec{\Phi}(t_0) + G_2(t_0)\overline{\vec{\Phi}(t_0)} + S^{(1)}\vec{\Phi} + \overline{S^{(2)}\vec{\Phi}} = \mathbf{F}(t_0), \quad (2)$$

where

$$S^{(l)}\vec{\Phi} \equiv \frac{1}{\pi i} \int_{\Gamma} \frac{K_l(t_0, t)\vec{\Phi}(t) dt}{t - t_0}, \quad G_l(t_0) = \left\| A_{\theta^{(n+\nu-j)}}^{(l)}(\alpha_j(t_0)) \right\|_{j,\nu=0}^{n-1},$$

$$K_l(t_0, t) = (t - t_0) \left\| \frac{K_{\theta^{(n+\nu-j)}}^{(l)}(\alpha_j(t_0), \alpha_{\nu}(t)) \alpha'_{\nu}(t)}{\alpha_{\nu}(t) - \alpha_{\nu}(t_0)} \right\|_{j,\nu=0}^{n-1} \quad (l = 1, 2),$$

$$\mathbf{F}(t_0) = \{f(t_0), f(\alpha(t_0)), \dots, f(\alpha_{n-1}(t_0))\}, \quad (3)$$

* The case when $\alpha(t)$ changes the orientation of Γ is considered in a completely analogous way.

** If $A_{\nu}^{(2)}(t_0) \equiv 0$, $K_{\nu}^{(2)}(t_0, t) \equiv 0$, then from (1) one obtains an equation that was studied in the author's works ^(1,2); equation (1) is studied in an analogous manner.

It is necessary to note that in the work of G. S. Litvinchuk ⁽³⁾, devoted to an equation of the form (1), an inaccuracy present in ⁽¹⁾ is pointed out and its correction is given. However, this correction (Proposition 2 of ⁽³⁾) is erroneous, as is illustrated by the example of the equation

$$\varphi(t) + \varphi(\alpha(t)) = 0, \quad \alpha(t) = t \exp\{2\pi i/3\}, \quad |t| = 1;$$

Proposition 2 is also false for an ordinary singular equation ($n = 1$).

The required correction to ⁽¹⁾ is contained in the author's work ⁽²⁾.

and $\vec{\Phi}(t) = \{\Phi_0(t), \dots, \Phi_{n-1}(t)\}$ is the unknown vector. It is easy to show that (1) and (2) are simultaneously solvable or not; moreover, if $\varphi(t)$ is a solution of (1), then the vector $\{\varphi(t), \varphi(\alpha(t)), \dots, \varphi(\alpha_{n-1}(t))\}$ satisfies (2), while if $\vec{\Phi}(t)$ is a solution of (2), then the function

$$n\omega(t) = \Phi_0(t) + \Phi_{n-1}(\alpha(t)) + \dots + \Phi_1(\alpha_{n-1}(t))$$

satisfies (1).

Along with the operator Λ , let us consider the operator Λ' conjugate to it:

$$\Lambda' \psi \equiv \sum_{\nu=0}^{n-1} \left\{ A_{n-\nu}^{(1)}(\alpha_{\nu}(t_0)) \alpha'_{\nu}(t_0) \psi(\alpha_{\nu}(t_0)) + \overline{A_{n-\nu}^{(2)}(\alpha_{\nu}(t_0)) \alpha'_{\nu}(t_0) t_0^2 \psi(\alpha_{\nu}(t_0))} \right. \\ \left. - \frac{1}{\pi i} \int_{\Gamma} \frac{K_{n-\nu}^{(1)}(t, t_0) \psi(t) dt}{\alpha_{n-\nu}(t) - t_0} - \frac{1}{\pi i} \int_{\Gamma} \frac{K_{n-\nu}^{(2)}(t, t_0) \overline{\psi(t)} dt}{\alpha_{n-\nu}(t) - t_0} \right\}$$

$$(A_n^{(l)}[\] \equiv A_0^{(l)}[\], \quad K_n^{(l)}[\] \equiv K_0^{(l)}[\]).$$

It is not hard to see that $(\Lambda')' \equiv \Lambda$. The system of equations associated with $\Lambda'\psi = 0$ will be

$$P_*\vec{\psi} \equiv A(t_0)\vec{\psi}(t_0) + t_0^2 \overline{B(t_0)\vec{\psi}(t_0)} - \frac{1}{\pi i} \int_{\Gamma} \frac{M(t, t_0)\vec{\psi}(t) dt}{t - t_0} - \frac{1}{\pi i} \int_{\Gamma} \frac{N(t, t_0)\overline{\vec{\psi}(t)} dt}{t - t_0} = 0,$$

where

$$\begin{aligned} A(t_0) &\equiv \left\| A_{\theta(n+\nu-j)}^{(1)}(\alpha_j(t_0)) \frac{\alpha'_j(t_0)}{\alpha'_\nu(t_0)} \right\|_{\nu, j=0}^{n-1}, \\ B(t_0) &\equiv \left\| A_{\theta(n+\nu-j)}^{(2)}(\alpha_j(t_0)) \frac{\alpha'_j(t_0)}{\alpha'_\nu(t_0)} \right\|_{\nu, j=0}^{n-1}, \\ M(t, t_0) &\equiv (t - t_0) \left\| \frac{K_{\theta(n+\nu-j)}^{(1)}(\alpha_j(t), \alpha_\nu(t_0)) \alpha'_j(t)}{\alpha_\nu(t) - \alpha_\nu(t_0)} \right\|_{\nu, j=0}^{n-1}, \\ N(t, t_0) &\equiv (t - t_0) \left\| \frac{K_{\theta(n+\nu-j)}^{(2)}(\alpha_j(t), \alpha_\nu(t_0)) \overline{\alpha'_j(t)}}{\alpha_\nu(t) - \alpha_\nu(t_0)} \right\|_{\nu, j=0}^{n-1}, \\ \vec{\psi}(t) &= \{\psi_0(t), \dots, \psi_{n-1}(t)\}. \end{aligned}$$

We shall call the operator Λ of normal type if

$$\Delta(t) \equiv \text{Det} \begin{vmatrix} G_1(t) - K_1(t, t) & G_2(t) + \overline{K_2(t, t)} \\ G_2(t) - K_2(t, t) & G_1(t) + K_1(t, t) \end{vmatrix} \neq 0 \quad (t \in \Gamma). \quad (4)$$

It is not difficult to verify that the operators Λ and Λ' are simultaneously of normal type or not. Below we shall everywhere assume that Λ is an operator of normal type.

Taking into account that, under condition (4), the Noether theorems (see (4,5)) are valid for system (2), it follows from what has been set forth that the numbers $k(\Lambda)$ and $k(\Lambda')$ are finite.

Let $\psi_j(t)$ ($j = 1, \dots, k(\Lambda')$) be a complete system of linearly independent solutions of the equation $\Lambda'\psi = 0$. Then the vectors

$$\mathbf{x}_j(t) = \{\psi_j(t), \alpha'(t)\psi_j(\alpha(t)), \dots, \alpha'_{r-1}(t)\psi_j(\alpha_{n-1}(t))\},$$

as is not hard to verify, will be solutions of the system $P'\mathbf{x} = 0$, conjugate to the system $P\vec{\Phi} = 0$. Hence, from what was said above, it immediately follows (cf. (2)) that, for the solvability of (1), it is necessary

it is necessary that

$$\operatorname{Re} \int_{\Gamma} f(t) \psi_j(t) dt = 0, \quad (5)$$

where $\psi_j(t)$ ($j = 1, 2, \dots, k(\Lambda')$) is a complete system of linearly independent solutions of the equation $\Lambda' \psi = 0$.

Let $\mathbf{x}_j(t) = \{x_0^j(t), \dots, x_{n-1}^j(t)\}$ ($j = 1, \dots, k(P')$) be a complete system of linearly independent solutions of the system $P' \mathbf{x} = 0$. Then, as is not difficult to verify, the functions

$$\omega_j(t) = x_0^j(t) + \frac{1}{\alpha'_{n-1}(\alpha(t))} x_{n-1}^j(\alpha(t)) + \dots + \frac{1}{\alpha'(\alpha_{n-1}(t))} x_1^j(\alpha_{n-1}(t))$$

$$(j = 1, 2, \dots, k(P'))$$

will be solutions of the equation $\Lambda' \psi = 0$. Hence, and from what has been set forth, it follows that (7) is sufficient for the solvability of (1) (cf. (2))*.

It is not difficult to show that $k(P) = k(P_*)$.

Denote by H the space of all vectors with n components satisfying the Hölder condition with some exponent μ , and by \tilde{H} the subspace of the space H whose elements are all vectors of the form (3).

Let $\vec{\Phi}^\delta(t) = \{\Phi_0^\delta(t), \dots, \Phi_{n-1}^\delta(t)\}$ ($\delta = 1, \dots, k(P)$) be a complete system of linearly independent solutions of the system $P_* \vec{\Phi} = 0$, and $\mathbf{x}^{(l)}(t) = \{x_0^l(t), \dots, x_{n-1}^l(t)\}$ ($l = 1, 2, \dots, k(P')$) a complete system of linearly independent solutions of the system $P' \mathbf{x} = 0$. Denote by $d \geq 0$ ($d' \geq 0$) the maximal number of vectors among $\{\vec{\Phi}^\delta(t)\}$ ($\{\mathbf{x}^{(l)}(t)\}$) satisfying the condition

$$\operatorname{Re} \int_{\Gamma} \mathbf{F}(t) \mathbf{x}(t) dt = 0 \quad \text{for all } \mathbf{F}(t) \in \tilde{H}. \quad (6)$$

Then from the results obtained it follows that $k(P') = k(\Lambda') + d'$, $k(P) = k(\Lambda) + d$, and, consequently, the following is valid.

Theorem 1. *The index of the operator Λ is computed by the formula*

$$\operatorname{ind} \Lambda = \frac{1}{\pi} \{\arg \Delta(t)\}_{\Gamma} - \chi(\Lambda), \quad (7)$$

where $\chi(\Lambda) = d - d'$.

Theorem 2. *If*

$$V\varphi \equiv \int_{\Gamma} n(t_0, t)\varphi(t) dt$$

is an arbitrary completely continuous operator acting in the Hölder space, then

$$\text{ind}(\Lambda + V) = \text{ind} \Lambda.$$

From the results obtained above it follows that every bounded and integrable solution of equation (1) satisfies the Hölder condition.

If the normality conditions (4) are violated, then, generally speaking, the equation $\Lambda\varphi = 0$ has an infinite set of linearly independent solutions. The case when $n = 2$, $A_0^{(1)} = A_1^{(2)} = K_0^{(1)} = K_1^{(2)} = 0$ and Γ is a simple closed Lyapunov contour was considered in (6).

In conclusion we note that the results obtained above are trivially generalized to the case of systems of equations of the form (1).

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* The theorem on the necessity and sufficiency of condition (7) is given in work (3).

Note: Figure translations are in progress. See original paper for figures.

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