THE METRIC OF SPACE-TIME AND NONLINEAR FIELDS

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Abstract

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THE METRIC OF SPACE-TIME AND NON-LINEAR FIELDS

At the basis of measuring space-time in the special theory of relativity (S.T.R.) lies the assumption of the constancy of the speed of light in the physical vacuum. Meanwhile this assumption, generally speaking, is not satisfied in nonlinear field theories ([1−4]), even though they may satisfy the formal requirements of Lorentz invariance.

Certain requirements on the linearity of a signal, implicitly contained in S.T.R., are usually overlooked. A. Einstein noted more than once that in fact we are given only the sum "geometry + physics," and not each term separately. Let us consider this "sum" in the example of an abstract world in which there exists only one kind of matter—a scalar field $\varphi(x)$, distributed in the space $R(x)$ $(x = x_1, x_2, x_3, x_4)$.

The metric of this space is initially not defined; however, we shall regard the law of distribution of the field $\varphi(x)$ as specified by means of the variational principle:

$$\delta \int L\, d\omega = 0, \tag{1}$$

where $d\omega = dx_1 dx_2 dx_3 dx_4$; L is the Lagrangian density, depending on the field $\varphi(x)$ and on its first derivatives $\varphi_i = \partial\varphi/\partial x_i$ $(i = 1, 2, 3, 4)$. From (1) there follows the field equation

$$a_{ik}\frac{\partial^2\varphi}{\partial x_i\, \partial x_k} + b_i\frac{\partial\varphi}{\partial x_i} + c = 0, \tag{2}$$

where

$$a_{ik} = +\partial^2 L/\partial x_i \partial x_k, \qquad b_i = \partial L/\partial\varphi\, \partial x_i, \qquad c = -\partial L/\partial\varphi, \tag{3}$$

are functions of φ and φ_i, so that equation (2), generally speaking, is nonlinear. To establish metric relations in the abstract world we are considering, we can use only the "physics" of the field $\varphi(x)$. If equation (2) is an equation of elliptic type, so that

$$a_{ik}\xi_i\xi_k > 0 \tag{4}$$

(ξ_i is an arbitrary real vector), then, in general, it is impossible to divide the set $R_4(x)$ into space and time. In this case a variation of the field $\delta\varphi(P)$ in the neighborhood of some point $P(x_1, x_2, x_3, x_4)$ leads to a change of the field φ in any region $\mathfrak{G}$ bounded by a closed surface Γ surrounding the point P. In this case none of the directions in $R_4(x)$ is distinguished, and the division into space and time is impossible (see Fig. 1).

If, however, equation (2) is an equation of hyperbolic type, so that

$$a_{ik}\xi_i\xi_k < 0, \tag{4'}$$

then a variation $\delta\varphi(P)$ in the neighborhood of the point $P(x_1, x_2, x_3, x_4)$ causes a variation of the field φ only at those points of $R_4(x)$ which lie inside the characteristic (cone of influence) formed by the characteristics of equation (2).

In this case the directions in the space $R_4(x)$ are divided into temporal (lying inside the cone of influence) and spatial (lying outside it). Along with this there also appears the concept of time; we can now speak of the propagation of a signal from the point P. Temporal points are reached by a signal from P, spatial ones are not.

Let us note that by a signal we mean a surface of weak discontinuity of the field $\varphi(x)$, $s = s(x_1, x_2, x_3, x_4) = \text{const}$. The family of these surfaces is determined from the equation (5)

$$a_{ik}\frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_k} = 0. \tag{5}$$

In view of the nonlinearity of equation (1), the characteristics of the equation (lines orthogonal to $s = \text{const}$) will depend on the magnitude of the field φ and its derivatives φ_i. Therefore the division of the set $R_4(x)$ into space and time will depend on the magnitude of the field φ and its derivatives.

Fig. 1. a—the case of an elliptic equation: in the set $R_2(t, x)$ there are no distinguished directions; b—the case of a hyperbolic equation: the set $R_2(t, x)$ is divided by the cone of influence cPc into time (shaded) and space (outside the cone cPc)

It is known that in a space with metric

$$ds^2 = g_{ik}dx_i dx_k \tag{6}$$

a signal (weak discontinuity) propagates according to the equation

$$g^{ik}\frac{\partial s}{\partial x_i}\frac{\partial s}{\partial x_k}=0, \tag{7}$$

where g^{ik} is the contravariant metric tensor (6).

Comparison of (7) and (5) shows that the metric in $R_4(x)$ must be consistent with the law of propagation of the signal (5). This consistency requires that

$$g^{ik}=\lambda a^{ik}, \tag{8}$$

where λ is a common scale factor. Thus the metric tensor g^{ik} will be a function of the field $\varphi(x)$ and its derivatives:

$$g^{ik}=g^{ik}(\varphi,\varphi_i). \tag{9}$$

The metric of $R_4(x)$ turns out to depend on the field φ.

Suppose now that, in addition to the field φ, there also exists another field ψ that likewise obeys an equation of the form (2). The field φ, if it is used as a signal for ordering events in $R_4(x)$, will determine another metric, different from (9). Which metric should we prefer?

It is clear that we must call time that region which is reached by any signal (ψ or φ). In other words, the division of $R_4(x)$ into space and time must be made with the help of that signal which has a cone of influence including all the other possible ones (see Fig. 2).

It is natural to assume that in the region of small fields the ordinary metric adopted in special relativity should hold. Therefore, far from the source P, the characteristics must straighten out, as shown in Fig. 3. This straightening can occur in two essentially different ways. The velocity of the signal may tend to the speed of light c from the side

lower velocities (in Fig. 3, the case φ) or from the side of greater velocities (in Fig. 3, the case ψ). In the latter case the metric of space-time must be redefined as applied to the law of propagation

Fig. 2. Characteristics of the field φ and ψ. The cone of influence lies inside the cone of influence of φ; therefore, in order to separate $R_2(t,x)$ into space and time, the signal of the field φ must be preferred

Fig. 3. The dashed line c denotes the cone of influence of s.t.o. (the speed of light in vacuum is taken to be constant). The cone with a metric adopted in s.t.o. The cone determined by the field ψ at small distances from P requires a change of the metric of s.t.o.

of this field near the point P. Thus, the field φ, which determines the metric of space, will interact with all other fields through the metric tensor

$$g^{ik} = g^{ik}(\varphi, \varphi_i).$$

Fig. 4. The dashed line shades the region of influence for the Klein equation with $m^2 < 0$. In this case the measurement x_4 should be regarded as spatial and the measurements x_1 (as well as x_2 and x_3) as temporal

In conclusion let us note that the Klein equation

$$\frac{\partial^2\varphi}{\partial x_4^2} - \left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}\right) - m^2\varphi = 0 \tag{10}$$

with negative $-m^2$ (“imaginary mass”), as is usually assumed, leads to the motion of particles with a velocity greater than the speed of light c. From the point of view of the definition of time described above (the region reached by the signal; see Fig. 4), in the case of equation (10) one should consider that we have one spatial coordinate and three temporal ones: x_1, x_2, x_3. The role of frequency should be played by the quantity $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$, so that $\omega^2 = k_4^2 + m^2$.

With such a definition of time, the velocities of particles described by equation (10) will be less than the speed of light c.

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