ON FRACTIONAL POWERS OF WEAKLY POSITIVE OPERATORS

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Abstract

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ON FRACTIONAL POWERS OF WEAKLY POSITIVE OPERATORS

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1. In ([1]) a new method was proposed for investigating fractional powers of a certain class of operators acting in the spaces L_p. It was established in which spaces L_{p_α} the operators $A^{-\alpha}$ act from L_{p_0}, if it is known in which L_{p_1} the operator A^{-1} acts from L_{p_0}. The proof is based on the moment inequality for fractional powers and the interpolation theorem from ([2]). In the present paper another method is proposed, which makes it possible to study the operators $BA^{-\alpha}$. The operators $BA^{-\alpha}$ were studied earlier in ([3]). However, restrictions were imposed there on the operator A, the validity of which in important cases of elliptic operators in domains has not yet been established.

2. An operator A, acting in a Banach space E, is called weakly positive (abbrev. w.p. E operator) if $D(A)$ is dense in E, the operator $A + tI$ has a bounded inverse for all $t \geqslant 0$, and $\|(A+tI)^{-1}\|_E \leqslant M(t+1)^{-1}$. Arbitrary powers of such operators are defined. In particular,

$$A^{-a} = \frac{\sin \pi(1-a)}{\pi(1-a)} \int_0^\infty t^{1-a}(A+tI)^{-2}\, dt \qquad (0 < a < 2). \tag{1}$$

The moment inequality is valid

$$\|A^\alpha v\|_E \leqslant \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} M_1^{1-\alpha} M_2^\alpha \|Av\|_E^\alpha \|v\|_E^{1-\alpha} \qquad (0 \leqslant \alpha \leqslant 1,\ v \in D(A)), \tag{2}$$

where

$$M_1 = \sup_{t>0} \|A(A+tI)^{-1}\|_E, \qquad M_2 = \sup_{t>0} \|t(A+tI)^{-1}\|_E.$$

Fractional powers of w.p. E operators were first introduced and studied in ([4]) (see also ([5])).

Theorem 1. Let A be a w.p. E operator and let $F(v)$ be a seminorm continuous on $D(A)$. Suppose that for some $\alpha \in (0,1)$ the inequality

$$F(v) \leqslant c\|Av\|_E^{\alpha}\|v\|_E^{1-\alpha} \qquad (v \in D(A)) \tag{3}$$

holds. Then for every $\beta \in (\alpha, 1]$ the seminorm $F(v)$ can be extended to a seminorm $\overline{F}(v)$ continuous on $D(A^{\beta})$, and for every $\gamma \in [0, \alpha)$ the inequality

$$\overline{F}(v) \leqslant c\,\frac{\beta-\gamma}{(\beta-\alpha)(\alpha-\gamma)}\,\frac{\sin\pi(1-\beta)}{\pi(1-\beta)}\left[\frac{\sin\pi(\beta-\gamma)}{\pi(\beta-\gamma)(1-\beta+\gamma)}\right]^{(\beta-\alpha)/(\beta-\gamma)} \times$$

$$\times M_1^{\alpha+(\beta-\alpha)/(\beta-\gamma)} M_2^{2-\alpha}\|A^{\beta}v\|_E^{(\alpha-\gamma)/(\beta-\gamma)}\|A^{\gamma}v\|_E^{(\beta-\alpha)/(\beta-\gamma)} \qquad (v \in D(A^{\beta})). \tag{4}$$

Proof. Using successively (1), (3), and (2), we obtain that for every $N > 0$ on $D(A)$ the inequality

$$F(v) \leqslant \frac{\sin\pi(1-\beta)}{\pi(1-\beta)}\int_0^{\infty} t^{1-\beta}F([A+tI]^{-1}A^{\beta}[A+tI]^{-1}v)\,dt \leqslant$$

$$\leqslant c\,\frac{\sin\pi(1-\beta)}{\pi(1-\beta)}M_1^{\alpha}M_2^{1-\alpha}\left[\int_0^{N} t^{\alpha-\gamma-1}\frac{\sin\pi(\beta-\gamma)}{\pi(\beta-\gamma)(1-\beta+\gamma)}M_1M_2\|A^{\gamma}v\|_E\,dt+\right.$$

$$\left.+\int_N^{\infty} t^{\alpha-\beta-1}M_2\|A^{\beta}v\|_E\,dt\right].$$

Minimizing the square bracket with respect to N, we obtain (4).

Remark 1. If for some $0 \leq \gamma < \alpha < \beta \leq 1$ (4) is satisfied, then (3) follows easily from (2).

It follows from (4) that $\overline{F}(A^{-\beta}v)$ is continuous in E and

$$\overline{F}(A^{-\beta}v) \leq c(\alpha-\gamma, \beta-\gamma)\|v\|_E^{(\alpha-\gamma)/(\beta-\gamma)}\|A^{-(\beta-\gamma)}v\|_E^{(\beta-\alpha)/(\beta-\gamma)}. \tag{5}$$

The estimate of $\|A^{-\alpha}v\|_E$ follows from (2), if an estimate of $\|A^{-1}v\|_E$ is known (see (1)). In a more general situation one applies

Theorem 2. For arbitrary $0 \leq \gamma < \alpha < \beta \leq 1$ the inequality

$$\|A^{-\alpha}v\|_E \leq \frac{\sin\pi\alpha}{\pi}\,\frac{\beta-\gamma}{(\beta-\alpha)(\alpha-\gamma)}[\varphi_{\beta-\beta}(v)]^{(\alpha-\gamma)/(\beta-\gamma)}[\varphi_{1-\gamma}(v)]^{(\beta-\alpha)/(\beta-\gamma)}, \tag{6}$$

$$\varphi_\delta(v) = \sup_{t>0} \|t^\delta (A+tI)^{-1} v\|_E.$$

Theorem 3. Let $\delta \in (0,1)$, and let E_δ be a Banach space containing E. Let A be a w.p. E_δ operator, and suppose the inequality

$$\|v\|_E \le c\|Av\|_{E_\delta}^\delta \|v\|_{E_\delta}^{1-\delta} \quad (v \in D(A) \subset E) \tag{7}$$

holds. Then the inequality

$$\varphi_{1-\delta}(v) \le c[M_1(E_\delta)]^\delta [M_2(E_\delta)]^{1-\delta} \|v\|_{E_\delta} \tag{8}$$

holds.

3. Below we consider w.p. $L_p(\Omega)$ operators A, where Ω is a domain of n-dimensional space with boundary S. Inequality (5) makes it possible to estimate $\overline{F}(A^{-\beta}v_e)$ on characteristic functions of measurable sets $e \subset \Omega$, if an estimate of $\|A^{-(\beta-\gamma)}v_e\|_{L_p}$ is known. It is easy to see that, in any case,

$$\|A^{-(\beta-\gamma)}v_e\|_{L_p} \le c(\operatorname{mes} e)^{1/p}.$$

Such estimates, for the particular case of seminorms $\|BA^{-\beta}v\|_{L_q}$, where $L_q = L_q(G)$, and G is a domain of m-dimensional space and B is an operator from $L_p(\Omega)$ into $L_q(G)$, together with the interpolation theorem from ([2]), make it possible to prove the boundedness of the operators $BA^{-\beta}$.

Theorem 4. Let A be a w.p. $L_{p_i}(\Omega)$ operator, $i = 1, 2$. Suppose that for some $\varepsilon_0, \delta_{0,i} \in (0,1)$

$$\|A^{-\varepsilon_0} v_e\|_{L_{p_i}} \le c_i (\operatorname{mes} e)^{\delta_{0,i}}. \tag{9}$$

Let B be a linear closed operator from $L_{p_i}(\Omega)$ into $L_{q_i}(G)$, $q_1 \ne q_2$, $D(B) \supset D(A)$, and

$$\|Bv\|_{L_{q_i}} \le D_i \|Av\|_{L_{p_i}}^{\alpha_0} \|v\|_{L_{p_i}}^{1-\alpha_0} \quad (v \in D(A)), \tag{10}$$

where α_0 is some number in $(0,1)$. Suppose β_0 satisfies the inequalities

$$\alpha_0 < \beta_0 < \min\{1, \alpha_0 + \varepsilon_0\}, \qquad \frac{1}{r_i} \equiv \frac{1}{p_i} + \frac{\beta_0 - \alpha_0}{\varepsilon_0}\left(\delta_{0,i} - \frac{1}{p_i}\right) \ge \frac{1}{q_i},$$

and the numbers q_t and r_t, for arbitrary $t \in (0,1)$, are defined by the equalities $1/q_t = t/q_1 + (1-t)/q_2$, $1/r_t = t/r_1 + (1-t)/r_2$. Then the operator $BA^{-\beta_0}$ admits a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

Hence follows

Theorem 5. Let A satisfy the conditions of Theorem 4, and let R be a w.p. $L_{q_i}(G)$ operator. Suppose T and RTA^{-1} are bounded operators from $L_{p_i}(\Omega)$ into $L_{q_i}(G)$. Then the operator $R^{\alpha_0}TA^{-\beta_0}$ admits a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

The proof is based on inequality (2), by means of which inequalities (10) are established for the operator $B = R^{\alpha_0}T$.

Remark 2. Let Q be a projection operator in $L_{p_i}(\Omega)$, and let A be a w.p. $E_i = QL_{p_i}(\Omega)$ operator. Then the operator $BA^{-\beta_0}Q$ from Theorem 4 and the operator $R^{\alpha_0}TA^{-\beta_0}Q$ from Theorem 5 admit a closure to a bounded operator from $L_{r_t}(\Omega)$ into $L_{q_t}(G)$.

4. Consider the boundary-value problem

$$Av + tv \equiv a(x; 1/iD)v(x) + (t_0 + t)v(x) = f(x) \quad (x \in \Omega), \tag{11}$$

$$B_{jv} \equiv B_j(x; 1/iD)v(x) = 0 \quad (x \in S,\ j = 1, \dots, k).$$

Let $a_0(x;\xi)$ and $B_{j0}(x;\xi)$ be the collections of the highest-order terms of the polynomials $a(x;\xi)$ and $B_j(x;\xi)$, of degrees $2k$ and m_j, respectively. Let $a_0(x;\xi) + t^{2k} \neq 0$ for every real vector ξ and $t > 0$. Let, for any $x \in S$, vectors ξ and η (tangent and normal to S at the point x), the polynomial

$$p(z) = a_0(x; \xi + z\eta) + t^{2k}$$

have k roots z_i^+ with $\operatorname{Im} z_i^+ > 0$. Let the polynomials $B_{j0}(x; \xi + z\eta)$ be linearly independent modulo the polynomial $(z - z_1^+) \cdots (z - z_k^+)$. Finally, let the system of operators B_j be normal ([13]), and $m_j \le 2k-1$. Then ([6,7]) the following holds.

Theorem 6. *For sufficiently large $t_0 > 0$ and for every $t \ge 0$, problem (11) is uniquely solvable in every $L_p(\Omega)$, and*

$$t\|v\|_{L_p} + \|v\|_{W_p^{2k}} \le c_p \|(A + tI)v\|_{L_p}. \tag{12}$$

From (12), first, it follows that A is a s.p. $L_p(\Omega)$ operator. Secondly, from (12) and the multiplicative inequalities for norms in the spaces of S. L. Sobolev–L. N. Slobodetskii (see ([8,9])) there follows the inequality

$$\|v\|_{W_p^{2k\gamma}} \le c\|Av\|_{L_{p_\delta}}^{\delta} \|v\|_{L_{p_\delta}}^{1-\delta} \quad (v \in D(A) \subset L_{p_\delta}(\Omega)) \tag{13}$$

for any

$$0 \le \gamma < 1, \quad \gamma < \delta < \min\left\{1, \frac{n}{2k}\left(1 - \frac{1}{p}\right) + \gamma\right\},$$

$$\frac{1}{p_\delta} = \frac{1}{p} + \frac{2k(\delta - \gamma)}{n}.$$

For $\gamma = 0$, from (13) we obtain an inequality of type (7), and for $\gamma \ge 0$, an inequality of type (10). When $2k\gamma = l + r$, l is an integer and $0 < r < 1$, one must consider the operator

$$Bv = |x - y|^{-n-r/p}[D_x^l v(x) - D_y^l v(y)],$$

acting from $D(A) \subset L_{p_\delta}(\Omega)$ into $L_p(\Omega \times \Omega)$. Thus one establishes

Theorem 7. *$A^{-\delta}$ acts from $L_{p_\delta}(\Omega)$ into $W_p^{2k\gamma}(\Omega)$ and is bounded.*

Only in the case when A is a self-adjoint operator in L_2 does the theory of scales ([10]) give a stronger result: the operator $A^{-\delta}$ acts from L_2 into $W_2^{2k\delta}$ and is bounded.

Remark 3. The conditions given above are satisfied by the operators A generated by the three classical boundary-value problems for elliptic operators of second order. The weak positivity of such operators was proved earlier in ([11]).

Let A be an operator acting in $\dot{H}_p$, generated by the stationary Navier–Stokes equations (see ([12])). From Theorems 7 and 5 (see Remark 2) it follows that

Theorem 8. *$A^{-\delta}$ acts from H_{p_δ} into $W_p^{2\gamma}$ and is bounded.*

5. The operator A^*, acting in $L_{p'}(\Omega)$ $(1/p + 1/p' = 1)$ and adjoint to the operator A acting in $L_p(\Omega)$, is generated by the adjoint differential expression a^* and the adjoint system of boundary conditions B_j^*, $j = 1, \dots, k$, which have the same properties as the differential operators a and B_j (see ([13])). Let $W_p^r(A)$ be the collection of functions $v(x) \in W_p^r(\Omega)$ satisfying the boundary conditions $B_{jv}(x) = 0$ for all $m_j \le r - 1$ (r integer). Define $W_{p'}^r(A^*)$ analogously.

Theorem 9. *Let $v \in W_p^{2k}(A)$, $u \in W_{p'}^r(A^*)$, $r = 0, 1, \dots, 2k$. Then*

$$\left| \int_\Omega Av\bar{u}\,dx \right| \le c(p, r)\|v\|_{W_p^{2k-r}}\|u\|_{W_{p'}^r}. \tag{14}$$

In the proof, constructions from ([13]) are used. Inequalities of the form (14) make it possible (see ([5,14,15])) to investigate $D(A^\alpha)$.

Theorem 10. *$W_p^{2k}(A) \supset D(A^\delta)$ in L_{p_δ} for integral $2k\gamma$, and $W_{p_\delta}^{2k\delta}(A) \subset D(A^\gamma)$ in L_p for integral $2k\delta$.*

Here $\gamma, \delta, p, p_\delta$ are the same as in (13). Only in the case when A is a self-adjoint operator in L_2, by means of the theory of scales [10], is a stronger result established: $D(A^\delta) = W_2^{2k\delta}(A)$ for integer $2k\delta$.

Theorem 11. Let l and m be integers, $0 \le l \le 2k-1,\ 1 \le m \le n$,

$$\frac{l}{2k} < \alpha < 1, \quad 0 < \delta < \left(\alpha - \frac{l}{2k}\right)\left(1 - \frac{l}{2k}\right)^{-1}, \quad \nu_\delta = \left(\alpha - \frac{l}{2k}\right) - \left(1 - \frac{l}{2k}\right)\delta,$$

$$p > \max\left\{1, \frac{n-m}{2k\nu\delta}\right\}, \quad \frac{1}{p_\delta} = \frac{1}{p}\frac{m}{n} + \frac{2k\nu_\delta}{n},$$

and let G be a smooth manifold of dimension m lying in Ω. Then

$$\|A^{-\alpha} v\|_{W_p^l(G)} \le c\, \|A^{-1}v\|^{\delta}_{W^l_{p_\delta}(\Omega)}\, \|v\|^{1-\delta}_{L_{p_\delta}(\Omega)}. \tag{15}$$

For the proof, in view of (1), one must estimate $\|(A+tI)^{-2}v\|_{W_p^l(G)}$ in two ways. First, from (13) there follows the estimate

$$\|(A+tI)^{-2}\|_{L_{p\delta}(\Omega) \to W_p^l(G)}.$$

Second,

$$\|(A+tI)^{-2}v\|_{W_p^l(G)} \le \|(A+tI)^{-1}\|_{L_{p\delta}(\Omega)\to W_p^l(G)} \|(A+tI)^{-1}v\|_{L_{p\delta}(\Omega)}.$$

The first factor is estimated by (13), and the second (see (14)) as follows:

$$\|(A+tI)^{-1}v\|_{L_{p\delta}(\Omega)} = \sup_u \left| \int_\Omega AA^{-1}v\,(A^*+tI)^{-1}u\,dx \right| \le$$

$$\le c\|A^{-1}v\|_{W^l_{p\delta}(\Omega)} \|(A^*+tI)^{-1}u\|_{W^{2k-l}_{p'_\delta}(\Omega)} \left(\frac{1}{p_\delta} + \frac{1}{p'_\delta} = 1, \quad \|u\|_{L_{p'_\delta}} = 1\right).$$

Next (13) is applied again.

Inequalities of type (15) make it possible to estimate $D^{lA^{-\alpha}}$, if an estimate is known for $D^{lA^{-1}}$. Attention is drawn to the latter circumstance in [3].

6. To clarify the question of which spaces L_{p_α} from L_{p_0} the operators $A^{-\alpha}$ act in, it is convenient to use the following rule. The operators $A^{-\alpha}$ are similar in this sense to integral operators with kernel $|x-y|^{2k\alpha-n}$ (for $\alpha > n/2k$); the operators $D^{lA^{-\alpha}}$ $(l < 2k\alpha)$ are similar to $A^{-\alpha+\beta}$, $\beta = l/2k$. For elliptic operators of second order this is not a formal rule, since it has been established [16] that their fractional powers and derivatives of fractional powers are integral operators with kernels of the indicated type. Apparently, this fact also holds for operators of arbitrary order $2k$.

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Note: Figure translations are in progress. See original paper for figures.