ON AUTOMORPHISMS OF THE ORDER STRUCTURE ON THE SET OF MATRIX NORMS

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Abstract

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ON AUTOMORPHISMS OF THE ORDER STRUCTURE ON THE SET OF MATRIX NORMS

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Let $\mathfrak{M}_p$ denote the ring of real square matrices of order p, and let $\mathfrak{N}_p$ denote the ordered (partially) set of norms* in $\mathfrak{M}_p$. The structure $\mathfrak{N}_p$ has already been considered in the works ([1,2]). Here we shall undertake its further study.

We shall call an automorphism of the structure $\mathfrak{N}_p$ a mapping of the set $\mathfrak{N}_p$ onto itself that preserves the order relation. An example of an automorphism is the mapping φ_u, which assigns to each norm $n(A)$ the norm $\varphi_u n(A) = n(UAU^{-1})$, and also the mapping φ_u^*: $\varphi_u^* n(A) = n(UA'U^{-1})$, where U is any nonsingular matrix and the prime denotes transposition.

Theorem. *The structure $\mathfrak{N}_p$ has no automorphisms other than φ_u and φ_u^*.*

Thus we have a complete description of the automorphism group of the structure $\mathfrak{N}_p$.

Before outlining the scheme of the proof, let us introduce one auxiliary notion, which will play a very important role in what follows.

Let $\pi \subseteq \mathfrak{M}_p$ be any subring. Denote by $\mathfrak{N}_p(\pi)$ the set of all functionals on π, each of which is the lower bound of some chain of norms on π. From the results of ([2]) it follows that $\mathfrak{N}_p(\mathfrak{M}_p) = \mathfrak{N}_p$.

Definition. Let $\pi \subseteq \mathfrak{M}_p$ be a subring and $\nu \in \mathfrak{N}_p(\pi)$. The **generalized norm** $n(A; \pi, \nu)$ $(A \in \mathfrak{M}_p)$ is the functional of A equal to ν on π and to infinity outside π. We shall denote the totality of all generalized norms by $\overline{\mathfrak{N}}_p$. The set $\overline{\mathfrak{N}}_p$ is considered ordered by the same principle as $\mathfrak{N}_p$.

Thus, generalized norms formally have all the properties of matrix norms, except, perhaps, positivity and finiteness on each matrix.

Lemma 1. *Let φ be any automorphism of the structure $\mathfrak{N}_p$. There exists, and moreover uniquely, an automorphism $\overline{\varphi}$ of the structure $\overline{\mathfrak{N}}_p$ that coincides with φ on $\mathfrak{N}_p$.*

In what follows we shall identify the automorphism $\overline{\varphi}$ with the automorphism φ. By virtue of Lemma 1, our theorem is equivalent to the analogous assertion for the structure $\overline{\mathfrak{N}}_p$. We shall prove precisely this assertion.

Consider the structure, ordered by inclusion, of the set of subrings of the ring $\mathfrak{M}_p$. The automorphism φ (henceforth fixed) induces a certain mapping Φ of this structure onto itself. Namely, if

$$\varphi n(A;\pi,\nu) \equiv n(A;\pi',\nu'),$$

then put $\Phi(\pi,\nu) = \pi'$. The images of one and the same subring π may be different subrings π', depending on the "parameter" ν. The mapping Φ, in a certain sense, preserves the relation–

* Recall that a norm in a ring is, by definition, "submultiplicative" :

$$\|AB\| \leq \|A\|\,\|B\|.$$

The order relation is defined in the natural way: if $\nu_1, \nu_2 \in \mathfrak{N}_p$, then

$$\nu_1 < \nu_2 \Longleftrightarrow \nu_1(A) \leq \nu_2(A), \quad (A \in \mathfrak{M}_p)\ \&\ \nu_1(A) \neq \nu_2(A).$$

the order of the subring structure. Namely, if $\pi_1 \subseteq \pi_2$ and $\nu_1(A) \geq \nu_2(A)$, $A \in \pi_1$, then $\Phi(\pi_1,\nu_1) \subseteq \Phi(\pi_2,\nu_2)$. The minimal elements of the subring structure will be the one-dimensional subrings G_X with one generator X such that $X^2 = \mu X$, and only these. Therefore it is clear that the set $G = \bigcup_X G_X$ is invariant with respect to Φ. Consequently, the set $\overline{\mathfrak{N}}_p^{(0)}$ of all norms of the form $n(A;G_X,\nu)$ is invariant with respect to the automorphism φ. In fact, one can prove somewhat more.

Lemma 2. The set $*\mathfrak{N}_X$ of all generalized norms of the form $n(A;G_X,\nu)$ for fixed X is carried by the automorphism φ into the set $\mathfrak{N}_Y$ of all norms of the form $n(A;G_Y,\nu)$ with some $Y = Y(\varphi,X)$.

In other words, $\Phi(G_X,\nu) = G_Y$ does not depend on ν. Thus, to each matrix $X \in G$, up to a factor, there corresponds a matrix $Y = \Phi_0(X) \in G$. Imposing on this correspondence the requirement of homogeneity: $\Phi_0(\lambda X) = \lambda\Phi_0(X)$, we obtain a single-valued operator Φ_0 on the set G. It is easy to see that if a subring $\pi \subset G$, then

$$\Phi(\pi,\nu) = \bigcup \Phi_0(X).$$

Lemma 3. If two norms $n_1, n_2 \in \overline{\mathfrak{N}}_p$ coincide on the matrix $X \in G$, then their images coincide on the matrix $\Phi_0(X)$.

Proof. It is clearly sufficient to prove the lemma for the case when $n_2 = n(A;G_X,\nu)$, $\nu = n_1(X)$. In this case $n_2 > n_1$, and therefore $\varphi n_2 > \varphi n_1 \equiv n_1'$. Suppose that $\varphi n_2(\Phi_0(X)) > n_1'(\Phi_0(X)) \equiv \nu'$. Consider the norm $n_2' =$

$n(A;G_{\Phi_0(X)},\nu')$. Then $\varphi n_2 > n_2' > n_1'$. Applying the inverse automorphism, we obtain: $n_2 > \varphi^{-1}n_2' > n_1$. At the same time certainly $n_2(X) > \varphi^{-1}n_2'(X) \geq n_1(X)$, contrary to the condition.

From Lemma 2 it follows that $\varphi n(A;G_X,\nu) = n(A;G_{\Phi_0(X)},\nu')$, where $\nu' = f(X,\nu) \geq 0$ is some monotone function in ν, finite everywhere. Moreover, the mapping Φ generates a certain automorphism of the structure of subrings lying in G. Investigation of this automorphism shows that it leaves invariant the set $\mathfrak{M}_p^{(1)} \subset G$ of matrices of rank one. Using Lemma 3, we arrive at the following result.

Lemma 4. There exists such a nonsingular matrix U that either

$$\Phi_0(T) = \varepsilon(T)U^{-1}TU \qquad (T \in \mathfrak{M}_p^{(1)}),\ |\varepsilon(T)| = 1,$$

or

$$\Phi_0(T) = \varepsilon(T)U^{-1}T'U \qquad (T \in \mathfrak{M}_p^{(1)}),$$

where $\varepsilon(T)$ is a scalar, $|\varepsilon(T)| = 1$. Moreover, $f(\nu,T) = \nu$ for $T \in \mathfrak{M}_p^{(1)}$.

From Lemma 4 there immediately follows the validity of our theorem for the subset of norms from $\mathfrak{N}_p^{(0)}$ that are finite on matrices of rank one. This, in turn, entails its validity for a fairly broad class of norms, first of all for those $n \in \overline{\mathfrak{N}}_p$ which are the structural lower bound of norms from the indicated subset. Further, we shall call a norm $n \in \overline{\mathfrak{N}}_p$ **minimal** with respect to a set $M \subset \mathfrak{N}_p$ if from the conditions: a) $n_1 \in \mathfrak{N}_p$, b) $n_1(A) = n(A)$ $(A \in M)$, it follows that c) $n_1 > n$. It is not difficult now to understand that our theorem is valid for norms minimal with respect to the set $\mathfrak{M}_p^{(1)}$ and, in particular (see (1)), for operator norms.

To complete the proof of the theorem, consider the subring D_V of all matrices of the form $Y = VXV^{-1}$, where X is a diagonal matrix and V is a fixed nonsingular matrix.

Lemma 5. The domain of finiteness of the norm $\varphi n(A;D_V,\nu)$ is the subring $D_{U^{-1}V}$ (or $D_{U^{-1}V'}$).

This assertion follows in an obvious way from the fact that any norm $n(A;D_V,\nu)$ is majorized and minorized by norms with the same domain of finiteness D_V, for which the validity of our theorem has already been established—

* It is a chain.

the theorem. Namely,

$$n_1 \equiv n(A;D_\nu,\nu_1) = \inf_{T \in D_\nu \cap \mathfrak{M}_p^{(1)}} n(A;G_T,\nu) \geq$$

$$\geq n(A; D_\nu, \nu) \geq n_2(A) \equiv n(A; D_\nu, \nu_2),$$

where $\nu_2(A) = r(A)$ is the spectral radius of the matrix $A \in D_\nu$. It is not difficult to verify that n_2 is minimal with respect to the set $\mathfrak{M}_p^{(1)}$, so that the theorem is valid for this norm.

Corollary. *If two norms $n_1, n_2 \in \overline{\mathfrak{M}}_p$ coincide on the subring D_ν, then their images coincide on the subring $D_{U^{-1}\nu}$ (or on $D_{U^{-1}\nu}$).*

The proof of this assertion does not differ in any way from the proof of Lemma 3.

From Lemma 5 and its corollary it follows that our theorem is valid for norms of the form $n(A; D_\nu, \nu)$.

Finally, let $n \in \mathfrak{M}_p$. From the equality $n(A) = \inf_\nu n(A; D_\nu, \nu_\nu)$, where $\nu_\nu(A) = n(A)$ $(A \in D_\nu)$, the validity of the theorem for the norm $n \in \mathfrak{M}_p$ follows, and consequently also for all norms in $\mathfrak{M}_p$.

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CITED LITERATURE

1. Yu. I. Lyubich, *UMN*, 18, No. 4, 161 (1963).
2. G. R. Belitskii, *DAN*, 151, No. 1, 9 (1963).

Note: Figure translations are in progress. See original paper for figures.