# ON ONE CASE OF AN INVERSE MIXED BOUNDARY-VALUE PROBLEM

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**Abstract**

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# ON ONE CASE OF AN INVERSE MIXED BOUNDARY-VALUE PROBLEM

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In the present work the following problem is posed and solved. The contour $L_z$ of a certain simply connected domain $D_z$ consists of polygons $L_z^j$ and unknown arcs $\widetilde{L}_z^j$, whose lengths are equal to $S_j$ $(j = 1, \dots, m)$. One of the vertices of the polygon $L_z^j$ is the point $z_1^j$. The length of the $i$-th side of the polygon $L_z^j$ is equal to $\lambda_j l_i^j$ $(i = 1, \dots, n_j - 1)$, where $\lambda_j$ is an unknown stretching coefficient $(j = 1, \dots, m)$.

On $\widetilde{L}_z^j$ two functions of the arc abscissa are prescribed,

$$u = f_1^j(s), \qquad v = f_2^j(s), \qquad s \in [0, S_j], \qquad j = 1, \dots, m, \tag{1}$$

which are the boundary values of an unknown function $w(z)$, analytic in the domain $D_z$. The boundary conditions for this function on the polygons $L_z^j$ are written in the form of the relations

$$\Phi_j(u, v) = 0, \qquad j = 1, \dots, m, \tag{2}$$

relating the real and imaginary parts of the sought function.

The boundary conditions (1) and (2) written above determine, in the plane of the complex potential $w$, the image $L_w$ of the contour $L_z$, bounding the domain $D_w$. The functions $f_k(s)$ and $\Phi_j(u, v)$ $(k = 1, 2;\ j = 1, \dots, m)$ are single-valued and twice continuously differentiable. Under the listed conditions we determine the contour $L_z$ and find a function $w(z)$, analytic in $D_z$, satisfying the boundary conditions (1) and (2).

The problem posed is a generalization of the case of one unknown arc $(m = 1)$ considered by V. N. Monakhov ([1]). Repeating, with natural complications, the solution given by him, we obtain the mixed boundary-value problem

$$\operatorname{Re} \ln \frac{dz}{dt} = \frac{ds}{dt} = \frac{\omega'(t)}{w_s'[s(t)]} = h_j(t) \quad \text{for } t \in [t_{n_j}^j, t_1^{j+1}], \tag{3}$$

$$\operatorname{Im} \ln \frac{dz}{dt} = \operatorname{arc\,tg} k_i^j \qquad \text{for } t \in [t_1^j, t_{n_j}^j].$$

Here $k_i^j$ is taken from the equation of the $i$-th side of the polygon $L_z^j$: $y = k_i^j x + b_i^j$; $\omega(t)$ are the boundary values of the function $\omega(\zeta)$, conformally mapping the upper half-plane of a certain $\zeta$-plane onto the domain $D_w$. From the correspondence of boundary points determined by the function $\omega(t)$, the images $t_1^j$ and $t_{n_j}^j$ of the ends of the polygons $L_z^j$ are found ($j = 1, \dots, m$; $n_j$ is the number of vertices of the polygon $L_z^j$). Three points $(t_1^1, t_{n_1}^1, t_1^2)$ are chosen so that the image of the infinitely distant point of the $t$-axis lies on one of the sides of the polygon $L_z^1$. The angular coefficient of the straight line on which this segment lies will be denoted by $\beta$.

Let $\alpha_1^j, \alpha_{n_j}^j$ be the angles of the contour $L_w$ at the points that are images of the ends of the polygon $L_z^j$ under the mapping $w(z)$ ($j = 1, \dots, m$). Then

$$h_j(t) = h_j^*(t)(t - t_1^j)^{\alpha_1^j - 1}(t - t_{n_j}^j)^{\alpha_{n_j}{}^{j-1}},$$

where $0 < h_j^*(t) < \infty$.

The solution of the boundary-value problem (3) has the form (compare with (1)):

$$z = F(\zeta) = \int_0^\zeta \Pi(\zeta) \exp M(\zeta)\, d\zeta + C, \tag{4}$$

where

$$\Pi(\zeta) = \Pi'(\zeta)(\zeta - t_1^j)^{1-\gamma_1^j+\alpha_1^j-1}(\zeta - t_{n_j}^j)^{1-\gamma_{n_j}{}^{j+\alpha^j}{}_{n_j}-1};$$

$$\Pi'(\zeta) = e^{i\beta} \prod_{j=1}^{m} \prod_{i=2}^{n_j-1} (\zeta - t_i^j)^{\alpha_i^j - 1}(\zeta - t_1^j)^{\gamma_1^j - 1}(\zeta - t_{n_j}^j)^{\gamma_{n_j}{}^{j-1}};$$

$$M(\zeta) = \frac{1}{\pi i} X(\zeta) \sum_{j=1}^{m} \left\{ \int_{t_{n_j}^j}^{t_1^{j+1}} X^{-1}(t) \ln \frac{n_j(t)}{\Pi'(t)} \frac{dt}{t - \zeta} + C_j + (\alpha_1^j - \gamma_1^j) \ln(\zeta - t_1^j) + (\alpha_{n_j}^j - \gamma_{n_j}^j) \ln(\zeta - t_{n_j}^j) \right\};$$

$$X(\zeta) = \prod_{j=1}^{m} \sqrt{\frac{\zeta - t_{n_j}^j}{\zeta - t_1^j}}.$$

In these formulas $t_i^j$ are the preimages of the vertices of the polygons; $\alpha_i^j$ ($j = 1, \dots, m;\ i = 2, \dots, n_j - 1$) are the angles at the vertices of the polygons; $\gamma_1^j, \gamma_{n_j}^j$ are the angles between the end sides of the polygons $L_z^j$ and the axis $OX$. The constants $C_j$ are chosen analogously to (1), in such a way that the function $M(\zeta)$ is bounded for any $\zeta$.

The solution (4) of the posed problem depends on a collection of constants, which can be determined from the system of equations

$$z_1^1 = F(t_1^1),$$

$$\lambda_j l_i^j = \int_{t_i^j}^{{t_{i+1}}^j} \left| \frac{dF}{dt} \right| dt; \qquad j = 1, \dots, m;\ i = 1, \dots, n_j - 1. \tag{5}$$

It is not difficult to see that system (5) completely determines the problem.

The proof of existence and uniqueness of the solution is carried out by the Weinstein continuity method (3). The most difficult part to prove, as is usual in problems connected with this method, is the proof of the lemma on local uniqueness—the main result of the present work. In the remaining part, the course of the proof is analogous to that given by V. N. Monakhov for the case of several unknown arcs, when the boundary values depend on the parameter $x = \operatorname{Re} z$ [2].

**Lemma (on local uniqueness of the solution).** If there exist constants $C_j, \lambda_j, t_i^j$ ($j = 1, \dots, m;\ i = 2, \dots, n_j - 1$) satisfying system (5), then the Jacobian of the system

$$\frac{D\left(x_1^1, y_1^1, l_1^1, \dots, l_{n_1-1}^1, \dots, l_1^m, \dots, l_{n_m-1}^m\right)}{D\left(t_2^1, \dots, t_{n_1-1}^1, \dots, t_2^m, \dots, t_{n_m-1}^m, \lambda_1, \dots, \lambda_m, C\right)}$$

at this point is different from zero.

For the proof of the lemma, we write the variations $\delta z_1^1$ and $\delta l_i^j$

$$\sum_{j=1}^{m} \sum_{i=2}^{n_j-1} \frac{\partial z_1^1}{\partial t_i^j} \, \delta t_i^j + \delta l = \delta z_1^1, \tag{6}$$

$$\sum_{j=1}^{m} \sum_{i=2}^{n_j-1} \frac{\partial l_k^\nu}{\partial t_i^j} \, \delta t_i^j + \frac{\partial l_k^\nu}{\partial \lambda_\nu} \, \delta \lambda_\nu = \delta l_k^\nu, \qquad \nu = 1, \dots, m; \quad k = 1, \dots, n_\nu - 1.$$

We shall show that the homogeneous system corresponding to system (6) has only the zero solution $\delta t_i^j = \delta\lambda_j = \delta C = 0$. Thus the lemma will, obviously, be proved.

Suppose the contrary. Then the variation of the function $\ln \dfrac{dz}{d\zeta}$ found from (4),

$$\delta \ln \frac{dz}{d\zeta} = \delta t_i^j \left\{ \sum_{j=1}^{m} \sum_{i=1}^{n_j} (1-\alpha_i^j)(\zeta - t_i^j)^{-1} + \frac{\partial M}{\partial t_i^j} \right\} \tag{7}$$

has a pole of first order at all points $t_i^j$ for which $\delta t_i^j \neq 0$ (the second term of equality (7) has no singularities at the points $t_i^j$, $i \neq 1, n_j$). At infinity these functions are bounded.

On the other hand, we shall find $\delta \ln \dfrac{dz}{d\zeta}$ from the boundary-value problem given below and compare it with what has been obtained.

Consider the function $\hat{z} = \dfrac{z(\zeta) - z(t_1^j)}{\lambda_j}$. For this function the polygon $\hat{L}_z^j$ remains in place under variation; for points $\hat{z}$ belonging to it, the function $\zeta(\hat{z})$ is real; consequently, in this case $\delta\zeta$ is real (as a function of $\zeta$). Since $\delta\hat{z} + \dfrac{d\hat{z}}{d\zeta}\delta\zeta = 0$ (see, for example, (2)), it follows that

$$\delta x = \operatorname{Re} \delta\hat{z} = -\frac{d\hat{x}}{dt}\delta\zeta, \qquad \delta y = \operatorname{Im} \delta\hat{z} = -\frac{d\hat{y}}{dt}\delta\zeta$$

for $t \in [t_1^j, t_{n_j}^j]$. Hence, after simple transformations, we find for $t \in [t_i^j, t_{i+1}^j]$

$$k_i^j \delta x - \delta y = \frac{\delta\lambda_j}{\lambda_j} \left[ b_i^j - k_i^j x(t_1^j) + y(t_1^j) \right] = \text{const},$$

where $j = 1, \dots, m$; $i = 1, \dots, n_j - 1$. After differentiating the last relation one can obtain

$$\arg \frac{\delta\, dz/dt}{dz/dt} = 0,$$

i.e.

$$\operatorname{Im} \delta \ln \frac{dz}{dt} = 0 \quad \text{for} \quad t \in \sum_{j=1}^{m} [t_1^j, t_{n_j}^j]. \tag{8}$$

Since $h_j(t)$ is not varied, we have

$$\operatorname{Re}\delta\ln\frac{dz}{dt}=0\quad\text{for}\quad t\in\sum_{j=1}^{m}[t_{n_j}^{j},t_1^{j+1}].\tag{9}$$

The solution of the homogeneous problem (8), (9), bounded at infinity (see (7)), has the form

$$\delta\ln\frac{dz}{d\zeta}=P_\nu(\zeta)\prod_{j=1}^{m}(\zeta-t_1^j)^{\pm1/2}(\zeta-t_{n_j}^j)^{\pm1/2},\tag{10}$$

where $P_\nu(\zeta)$ is a polynomial of degree $\nu$ $(0\le\nu\le m)$, and the signs of the exponents depend on the boundedness of the solution at the corresponding point.

Comparing (10) and (7), we see that all $\delta t_i^j=0$, since, by formula (10), $\delta\ln\dfrac{dz}{d\zeta}$ has no singularities at the points $t_i^j$ $(j=1,\dots,m;\ i=2,\dots,n_j-1)$. From (6) and (5) we see that in this case $\delta C=\delta\lambda_j=0$. The lemma is proved.

The problem is considered analogously when the lengths of the sides of the polygons $L_z^j$ are known, while the lengths of the unknown arcs are not prescribed. The functions $f_1^j$ and $f_2^j$ depend ⋯

depend on the parameter $\psi=s/S_j$, where $s\in[0,S_j]$, $S_j$ is unknown. The lemma on local uniqueness remains valid.

The function $w(z)$ is found by the formula

$$w(z)=\omega[F^{-1}(z)],\tag{11}$$

where $\zeta=F^{-1}(z)$ is the function inverse to $z=F(\zeta)$, determined by equality (4).

In conclusion, we shall give one of the possible applications of the results obtained to the problem of antenna synthesis at ultrahigh frequencies.

As is known, one of the contemporary urgent problems is the synthesis of antennas from prescribed radiation patterns. The radiation pattern of an antenna depends on the geometry of the latter and on the distribution of current on it. We shall denote the function of the current distribution over the surface of the antenna by $F(x,y)$, where $x$ and $y$ are the linear dimensions of the antenna, and $F(x,y)$ is the amplitude of the current at the point $(x,y)$. It is known from radio engineering that $F(x,y)$ is a harmonic function of its variables.

Suppose that, from theoretical, design, or technological considerations, the shape of part of the antenna contour is specified up to a stretching, and that the rates $F'_x$ and $F'_y$ of change of the current as a function of the change of the linear dimensions on this part of the contour satisfy a certain relation. Suppose also that the rates $F'_x$ and $F'_y$ of change of the current are prescribed on the unknown part of the contour (for example, as functions of the distance traveled along the unknown part of the arc from the end point of the known part of the antenna contour). Under these conditions the inverse mixed boundary-value problem makes it possible to determine completely the antenna contour and to reconstruct the function of the current distribution over its surface. Indeed, for the analytic function $\Phi = F'_x - iF'_y$, one obtains the inverse mixed boundary-value problem considered above.

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*