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Abstract

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ON THE NIELSEN–TSENOV EQUATIONS AND THEIR APPLICATION TO NONHOLONOMIC SYSTEMS WITH NONLINEAR CONSTRAINTS

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1. We start from the identity ([1])

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_\chi} = \frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q}_\chi} - \frac{\partial T}{\partial q_\chi}\right) \quad (\chi = 1, 2, \ldots, k),$$

$$\overset{(n)}{T} = \frac{d^n T}{dt^n}, \qquad \overset{(n)}{q}_\chi = \frac{d^n q_\chi}{dt^n} \quad (n = 1, 2, \ldots), \tag{1}$$

where $T = T(t, \ldots, q_\chi, \ldots, \dot q_\chi, \ldots)$ is the kinetic energy; $q_\chi, \dot q_\chi$ are, respectively, the generalized coordinates and generalized velocities of the rheonomic-holonomic mechanical systems under consideration with k degrees of freedom, or from the generalized d' Alembert principle ([2])

$$\sum_{\nu=1}^{N} (\mathbf{F}_\nu - m_\nu \mathbf{w}_\nu) \cdot \delta \overset{(n)}{\mathbf{r}}_\nu = 0,$$

$$\delta t = 0, \qquad \delta \mathbf{r}_\nu = \delta \dot{\mathbf{r}}_\nu = \delta \ddot{\mathbf{r}}_\nu = \cdots = \delta \overset{(n-1)}{\mathbf{r}}_\nu = 0, \qquad \delta \overset{(n)}{\mathbf{r}}_\nu \neq 0, \tag{2}$$

where $\mathbf{F}_\nu$ is the resultant of the active forces acting on the ν-th material point P_ν with mass m_ν. We obtain ([1–3]) the generalized Lagrange equation

$$\frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q}_\chi} - (n+1)\frac{\partial T}{\partial q_\chi}\right) = Q_\chi \quad (\chi = 1, 2, \ldots, k;\ n = 1, 2, \ldots), \tag{3}$$

where Q_χ is the generalized force corresponding to q_χ.

For $n = 1$ we obtain the Nielsen equations ([4,5])

$$\frac{\partial \dot{T}}{\partial \dot{q}_\chi} - 2\frac{\partial T}{\partial q_\chi} = Q_\chi \quad (\chi = 1, 2, \dots, k); \tag{4}$$

for $n = 2$, the Tsenov equations ([6])

$$\frac{1}{2}\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_\chi} - 3\frac{\partial T}{\partial q_\chi}\right) = Q_\chi \quad (\chi = 1, 2, \dots, k). \tag{5}$$

Equations (4) also follow directly from the Jourdain principle ((2), $n = 1$), and equations (5) from the Gauss principle ((2), $n = 2$). It is shown directly that the Nielsen equations are obtained at once also from the Lagrange equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\chi} - \frac{\partial T}{\partial q_\chi} = Q_n \quad (\chi = 1, 2, \dots, k), \tag{6}$$

associated with the d' Alembert-Lagrange principle ((2), $n = 0$). For this purpose it is sufficient to differentiate $\partial T/\partial \dot{q}_\chi$ ([5]).

2. Suppose that in the function T the generalized velocities $\dot{q}_\chi$ are fixed, and in this case denote T by T_0; then

$$\frac{\partial T}{\partial q_\chi} = \frac{\partial \dot{T}_0}{\partial \dot{q}_\chi} = \frac{\partial \ddot{T}_0}{\partial \ddot{q}_\chi} = \cdots \qquad (\chi = 1, 2, \dots, k). \tag{7}$$

Following Tzenov' s method ([6]), using the transformations

$$R_1 = T - 2\dot{T}_0, \qquad \text{respectively} \qquad R_2 = \frac{1}{2}(\ddot{T} - 3\ddot{T}_0), \tag{8}$$

we reduce equations (4) and (5) to the equations

$$\frac{\partial R_1}{\partial \dot{q}_\chi} = Q_\chi, \qquad \text{respectively} \qquad \frac{\partial R_2}{\partial \ddot{q}_\chi} = Q_\chi \qquad (\chi = 1, 2, \dots, k), \tag{9}$$

analogous to Appell' s equations

$$\frac{\partial S}{\partial \ddot{q}_\chi} = Q_\chi \qquad (\chi = 1, 2, \dots, k), \tag{10}$$

(S is the energy of acceleration), valid both for holonomic and for nonholonomic mechanical systems with linear constraints.

With the aid of Appell’ s substitution (7)

$$K_1 = R_1 - \sum_{\chi=1}^{k} Q_\chi \dot{q}_\chi, \qquad \text{respectively} \qquad K_2 = R_2 - \sum_{\chi=1}^{k} Q_\chi \ddot{q}_\chi \tag{11}$$

we obtain

$$\frac{\partial Q_\chi}{\partial \dot{q}_\chi} = \frac{\partial Q_\chi}{\partial \ddot{q}_\chi} = 0 \qquad (\chi = 1, 2, \dots, k),$$

$$\frac{\partial K_1}{\partial \dot{q}_\chi} = 0, \qquad \text{respectively} \qquad \frac{\partial K_2}{\partial \ddot{q}_\chi} = 0, \tag{12}$$

coinciding with the equations to which the problem of finding the stationary values of the functions K_1 and K_2 is reduced.

3. Turning to nonholonomic systems with linear constraints with respect to the generalized velocities, respectively the generalized accelerations, we consider the expressions

$$\sum_{\chi=1}^{k} a'_{\rho\chi} \dot{q}_\chi + a'_\rho = 0, \qquad \text{respectively} \qquad \sum_{\chi=1}^{k} a''_{\rho\chi} \ddot{q}_\chi + a''_\rho = 0 \tag{13}$$

$$(\rho = 1, 2, \dots, r < k),$$

where $a'_{\rho\chi}$ and a'_ρ are functions only of time t and of the generalized coordinates q_χ, while $a''_{\rho\chi}$ and a''_ρ are also functions of the generalized velocities $\dot{q}_\chi$. Solving (13) with respect to $\dot{q}_\rho$ and $\ddot{q}_\rho$, we obtain the expressions

$$\dot{q}_\rho = \sum_{\lambda=1}^{l} a'_{\rho\lambda} \dot{q}_\lambda + a'_\rho, \qquad \text{respectively} \qquad \ddot{q}_\rho = \sum_{\lambda=1}^{l} a''_{\rho\lambda} \ddot{q}_\lambda + a''_\rho \tag{14}$$

$$(\rho = l + 1, l + 2, \dots, l + r = k),$$

which we substitute into the functions $K_1 = K_1(\dot{q}_\chi)$, respectively $K_2 = K_2(\ddot{q}_\chi)$. Thus we obtain the functions K_1^*, respectively K_2^*, depending only on $\dot{q}_\lambda$, respectively on $\ddot{q}_\lambda$. The required equations of motion now take the form

$$\frac{\partial K_1^*}{\partial \dot{q}_\lambda} = 0, \qquad \frac{\partial K_2^*}{\partial \ddot{q}_\lambda} = 0, \qquad (\lambda = 1, 2, \dots, l), \tag{15}$$

or, by virtue of (14), the form (8)

$$\frac{\partial K_1}{\partial \dot{q}_\lambda} + \sum_{\rho=l+1}^{l+r} \frac{\partial K_1}{\partial q_\rho} a'_{\rho\lambda} = 0, \qquad \frac{\partial K_2}{\partial \ddot{q}_\lambda} + \sum_{\rho=l+1}^{l+r} \frac{\partial K_2}{\partial q_n} a''_{\rho\lambda} = 0 \tag{16}$$

$$(\lambda = 1, 2, \dots, l).$$

4. Suppose that the constraints are nonlinear with respect to the generalized velocities, respectively to the generalized accelerations, and have the form

$$\Phi'_\rho(t, \dots, q_\chi, \dots, \dot{q}_\chi, \dots) = 0, \qquad \Phi''_\rho(t, \dots, q_\chi, \dots, \dot{q}_\chi, \dots, \ddot{q}_\chi \dots) = 0 \tag{17}$$

$$(\chi = 1, 2, \dots, k;\ \rho = 1, 2, \dots, r < 2).$$

In this case one may proceed in two ways. In the first case we repeat the transformations carried out in § 3, for which it is necessary to differentiate (17) with respect to t, namely

$$\sum_{\chi=1}^{k} \frac{\partial \Phi'_\rho}{\partial \dot{q}_\chi} \ddot{q}_\chi + \dots = 0, \qquad \sum_{\chi=1}^{k} \frac{\partial \Phi''_\rho}{\partial \ddot{q}_\chi} \dddot{q}_\chi + \dots = 0 \quad (\rho = 1, 2, \dots, r). \tag{18}$$

Equations (18) are already linear with respect to $\ddot{q}_\chi$, respectively $\dddot{q}_\chi$. Then one must again use Tzenov' s equations (5), respectively equations (3) for $n = 3$, i.e. (9),

$$\frac{1}{3}\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_\chi} - 4\frac{\partial \dot{T}}{\partial q_\chi}\right) = Q_\chi \quad (\chi = 1, 2, \dots, k), \tag{19}$$

and the transformations

$$R_3 = \frac{1}{3}\left(\ddot{T} - 4\ddot{T}_0\right), \qquad K_3 = R_3 - \sum_{\chi=1}^{k} Q_\chi \dddot{q}_\chi \quad (\partial Q_\chi / \partial \dddot{q}_\chi = 0). \tag{20}$$

In the second case one must apply Tzenov' s method, but not only for second-kind constraints nonlinear (with respect to velocity), but also for first-kind constraints; in the latter case one has again to use not Tzenov' s equations, but Nielsen' s equations. For this, let us differentiate the constraints (17). Eliminating from

$$\sum_{\chi=1}^{k} \frac{\partial \Phi'_\rho}{\partial \dot{q}_\chi} d\dot{q}_\chi + \dots = 0, \qquad \sum_{\chi=1}^{k} \frac{\partial \Phi''_\rho}{\partial \ddot{q}_\chi} d\ddot{q}_\chi + \dots = 0 \quad (\rho = 1, 2, \dots, r), \tag{21}$$

$$dK_1 \equiv \sum_{\chi=1}^{k} \frac{\partial K_1}{\partial \dot{q}_\chi} d\dot{q}_\chi + \cdots = 0, \qquad dK_2 \equiv \sum_{\chi=1}^{k} \frac{\partial K_2}{\partial \ddot{q}_\chi} d\ddot{q}_\chi + \cdots = 0 \tag{22}$$

respectively r differentials $d\dot{q}_\chi$ and r differentials $d\ddot{q}_\chi$, we finally obtain the equations

$$\frac{\partial K_1^*}{\partial \dot{q}_\lambda} d\dot{q}_\lambda = \left(\frac{\partial K_1}{\partial \dot{q}_\lambda} + \sum_{\rho=l+1}^{l+r} \frac{\partial K_1}{\partial q_\rho} \gamma'_{\rho\lambda} \right) d\dot{q}_\lambda = 0 \quad (\lambda = 1, 2, \dots, l), \tag{23}$$

$$\frac{\partial K_2^*}{\partial \ddot{q}_\lambda} d\ddot{q}_\lambda = \left(\frac{\partial K_2}{\partial \ddot{q}_\lambda} + \sum_{\rho=l+1}^{l+r} \frac{\partial K_2}{\partial q_\rho} \gamma''_{\rho\lambda} \right) d\ddot{q}_\lambda = 0 \quad (\lambda = 1, 2, \dots, l). \tag{24}$$

Consequently, for the constraints encountered in practice, linear and nonlinear with respect to the generalized velocities $\dot{q}_\chi$, the equations of motion have the form of Nielsen's equations of motion (4), derived from Jourdain's principle, and for constraints linear and nonlinear with respect to the generalized accelerations $\ddot{q}_\chi$, the form of Tzenov' s equations (5), derived from Gauss' s principle and (19).

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Note: Figure translations are in progress. See original paper for figures.