# ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF $E$-FUNCTIONS SATISFYING LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

MATHEMATICS

1966

SovietRxiv

## Abstract

## Full Text

UDC 511.84

*MATHEMATICS*

**A. B. SHIDLOVSKII**

# ON THE TRANSCENDENCE AND ALGEBRAIC INDEPENDENCE OF VALUES OF $E$-FUNCTIONS SATISFYING LINEAR NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

*(Presented by Academician A. Yu. Ishlinskii, 18 X 1965)*

In 1929 C. Siegel ([1,2]) considered the functions

$$K_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda+1)\dots(\lambda+n)} \left(\frac{z}{2}\right)^{2n}, \qquad \lambda \neq -1, -2, \dots,$$

which are a solution of the differential equation

$$y'' + \frac{2\lambda+1}{z} y' + y = 0. \tag{1}$$

He proved that if $\lambda$ is a rational number not equal to half an odd number, then $K_\lambda(\alpha)$ and $K'_\lambda(\alpha)$ are algebraically independent for any algebraic $\alpha \neq 0$.

In 1962 V. A. Oleinikov ([3]) considered the Kummer functions

$$A_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{\nu(\nu+1)\dots(\nu+n-1)}{n!\lambda(\lambda+1)\dots(\lambda+n-1)} z^n, \qquad \lambda, \nu \neq 0, -1, -2, \dots,$$

satisfying the differential equation

$$y'' + \left(\frac{\lambda}{z} - 1\right) y' - \frac{\nu}{z} y = 0, \tag{2}$$

and proved that if $\lambda$ and $\nu$ are rational numbers, $\nu - \lambda \neq 0, 1, 2, \dots$, then the numbers $A_{\lambda,\nu}(\alpha)$ and $A'_{\lambda,\nu}(\alpha)$ are algebraically independent for any algebraic $\alpha \neq 0$.

In 1954 the author ([4,5]) considered the functions

$$K_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda+1)\ldots(\lambda+n)(\mu+1)\ldots(\mu+n)} \left(\frac{z}{2}\right)^{2n},$$

which are a solution of the differential equation

$$y'' + \frac{2\lambda+2\mu+1}{z} y' + \left(1 + \frac{4\lambda\mu}{z^2}\right) y = \frac{4\lambda\mu}{z^2}, \tag{3}$$

and the following was established.

**Theorem 1.** *If $\lambda$ and $\mu$ are rational numbers distinct from negative integers, for which the difference $\lambda - \mu$ is not equal to half an odd number, and $\alpha \neq 0$ is any algebraic number, then the numbers $K_{\lambda,\mu}(\alpha)$ and $K'_{\lambda,\mu}(\alpha)$ are algebraically independent.*

The proof of this theorem was complicated and long, since the method of proving algebraic independence of functions, used by Siegel for solutions of linear homogeneous differential equations, could not be applied to solutions of nonhomogeneous equations. But it turns out that in a number of cases this method can be applied to solutions

of nonhomogeneous equations by means of a certain passage to the solutions of the corresponding homogeneous equations. Thanks to this, the proof of Theorem 1 is easily reduced to already proved facts, and new theorems are also obtained on the algebraic independence of values of other functions satisfying linear nonhomogeneous differential equations. We shall present the indicated passage in the form of a lemma generalizing the corresponding lemma of Siegel ([1,2]).

**Lemma 1.** Suppose

$$y'_k = A_{k,0} + \sum_{i=1}^{m} A_{k,i} y_i, \quad i = 1, \ldots, m; \quad m \geqslant 2, \tag{4}$$

is a system of $m$ linear (nonhomogeneous or homogeneous) differential equations of the first order, all coefficients $A_{k,i}$ of which belong to a given field $\mathcal{L}$ of analytic functions of $z$, closed under the operation of differentiation, and this system has a solution $y_1^0, \ldots, y_m^0$ such that the functions $y_1^0, \ldots, y_{m-1}^0$ are algebraically independent over the field $\mathcal{L}$, but

$$P(y_1^0, \ldots, y_m^0) = 0, \tag{5}$$

where $P(x_1, \ldots, x_m)$ is an irreducible polynomial in $m$ variables with coefficients from $\mathcal{L}$; $Q(x_1, \ldots, x_m)$ is the aggregate of its homogeneous terms of highest degree in $x_1, \dots, x_m$. Then there exists a solution $y_1^*, \dots, y_m^*$ of the system of $m$ linear homogeneous equations

$$y'_k = \sum_{i=1}^{m} A_{k,i} y_i, \quad i = 1, \dots, m, \tag{6}$$

corresponding to system (4), satisfying the condition

$$Q(y_1^*, \dots, y_m^*) = 0. \tag{7}$$

**Proof.** Let $y_1, \dots, y_m$ be an arbitrary solution of system (4). Consider the polynomial $P = P(y_1, \dots, y_m)$. If one computes its total derivative with respect to $z$, then this derivative $\frac{d}{dz}P$ will be a polynomial in $y_1, \dots, y_m, y'_1, \dots, y'_m$ with coefficients from $\mathcal{L}$. Replace in $\frac{d}{dz}P$ the variables $y'_1, \dots, y'_m$ by the right-hand sides of the corresponding equations of system (4). Then $\frac{d}{dz}P$ will be a polynomial $P' = P'(y_1, \dots, y_m)$ in $y_1, \dots, y_m$ with coefficients from $\mathcal{L}$. The polynomial $P'$ must be divisible by the irreducible polynomial $P$ as a polynomial in $m$ independent variables. Indeed, in view of (5) the equation

$$P'(y_1^0, \dots, y_m^0) = 0 \tag{8}$$

holds. If $P'$ were not divisible by $P$, then, eliminating the variable $y_m^0$ from the two equations (5) and (8), we would obtain an algebraic equation between $y_1^0, \dots, y_{m-1}^0$ with coefficients from $\mathcal{L}$, which is impossible. But the degrees of $P$ and $P'$ with respect to the aggregate $y_1, \dots, y_m$ are equal. Therefore the quotient obtained by dividing $P'$ by $P$ will be a function $A$ from $\mathcal{L}$. Hence, identically in $y_1, \dots, y_m$,

$$P'(y_1, \dots, y_m) = AP(y_1, \dots, y_m). \tag{9}$$

Denote by $Q'(y_1, \dots, y_m)$ the aggregate of the homogeneous terms of highest degree in $y_1, \dots, y_m$ occurring in $P'$. Then from (9) we have, identically in $y_1, \dots, y_m$,

$$Q'(y_1, \dots, y_m) = AQ(y_1, \dots, y_m). \tag{10}$$

Let $y_1^*, \dots, y_m^*$ be an arbitrary solution of system (6). Consider the polynomial $P^* = P(y_1^*, \dots, y_m^*)$. If one computes its total derivative with respect to $z$ and then replaces $y_1^{*\prime}, \dots, y_m^{*\prime}$ by the right-hand sides of the corresponding equations of system (6), then this derivative will also be some polynomial $P^{*\prime} = P^{*\prime}(y_1^*, \dots, y_m^*)$ with coefficients from $\mathcal{L}$, not coinciding with $P'$ in the case of the nonhomogeneity of system (4). But in passing from the polynomials $P$ and $P^*$ to the polynomials $P'$ and $P^{*\prime}$, the collections of homogeneous terms of highest degree in the latter are formed in the same way, in view of the coincidence of system (6) with the homogeneous part of system (4). Therefore the collection of homogeneous terms of highest degree in $P^{*\prime}$ will be $Q'(y_1^*, \dots, y_m^*)$, and this means that

$$\frac{d}{dz} Q(y_1^*, \dots, y_m^*) = Q'(y_1^*, \dots, y_m^*), \tag{11}$$

since, in passing from $P^*$ to $P^{*\prime}$, the homogeneous terms of each degree pass into homogeneous terms of the same degree, and in view of (10) we have

$$Q'(y_1^*, \dots, y_m^*) = AQ(y_1^*, \dots, y_m^*). \tag{12}$$

Consider the differential equation

$$v' = Av. \tag{13}$$

Its general solution is $v = cv_0$, where $c$ is a constant, and $v_0 \neq 0$ is some particular solution. From (11) and (13) it follows that the function $Q(y_1^*, \dots, y_m^*)$ is a solution of equation (13) for any solution $y_1^*, \dots, y_m^*$ of system (6). Let $y_{11}^*, \dots, y_{m1}^*$ and $y_{12}^*, \dots, y_{m2}^*$ be any two fixed linearly independent solutions of system (6). Then $y_k^* = \lambda_1 y_{k1}^* + \lambda_2 y_{k2}^*$, $k = 1, \dots, m$, where $\lambda_1$ and $\lambda_2$ are arbitrary constants, is also a solution of this system. Therefore

$$Q(\lambda_1 y_{11}^* + \lambda_2 y_{12}^*, \dots, \lambda_1 y_{m1}^* + \lambda_2 y_{m2}^*) = C(\lambda_1, \lambda_2) v_0, \tag{14}$$

where $C(\lambda_1, \lambda_2)$ is a homogeneous polynomial in $\lambda_1$ and $\lambda_2$ with constant coefficients. In view of this, one can choose $\lambda_1$ and $\lambda_2$, not both equal to zero, so that $C(\lambda_1, \lambda_2) = 0$. Then, denoting the corresponding solution $y_k^* = \lambda_1 y_{k1}^* + \lambda_2 y_{k2}^*$, $k = 1, \dots, m$, from (14) we obtain equation (7).

**Lemma 2.** Suppose all coefficients of the linear (nonhomogeneous or homogeneous) differential equation

$$y^{(m)} + A_{m-1} y^{(m-1)} + \dots + A_1 y' + A_0 y = B, \qquad m \geq 2, \tag{15}$$

belong to a given field $\mathcal{L}$ of analytic functions of $z$, closed under the operation of differentiation, and this equation has a solution $y_0$ that satisfies no algebraic differential equation of order less than $m-1$ with coefficients in $\mathcal{L}$, and for $m = 2$ is not an algebraic function over $\mathcal{L}$, but

$$P(y_0, y_0', \dots, y_0^{(m-1)}) = 0,$$

where $P(x_1, \dots, x_m)$ is an irreducible polynomial in $m$ variables with coefficients in $\mathcal{L}$, and $Q(x_1, \dots, x_m)$ is the collection of its homogeneous terms of highest degree in $x_1, \dots, x_m$. Then there exists a solution $y^*$ of the linear homogeneous differential equation corresponding to equation (15), satisfying the condition

$$Q(y^*, y^{*\prime}, \dots, y^{*(m-1)}) = 0.$$

In particular, for $m = 2$, $y_0$ is such that its logarithmic derivative is an algebraic function over $\mathcal{L}$.

Lemma 2 evidently follows from Lemma 1.

**Lemma 3** (see ([1,2])). If $\lambda$ is a complex number not equal to half an odd number, and $y \neq 0$ is any solution of the differential equation (1), then $y$ satisfies no algebraic differential equation of the first order with polynomial coefficients.

**Proof of Theorem 1.** Consider the function $y_0 = z^{2\mu} K_{\lambda,\mu}(z)$, satisfying the equation

$$y'' + \frac{2(\lambda - \mu) + 1}{z} y' + y = 4\lambda\mu z^{\mu-2}.$$

Since $K_{\lambda,\mu}(z)$ is an entire function, $y_0$ is not an algebraic function. If one assumes that $y_0$ satisfies an algebraic differential equation of the first order, then, by Lemma 2, we obtain that equation (1) has a solution $y^*$ whose logarithmic derivative is an algebraic function. But this contradicts Lemma 3. Consequently, $y_0$ and $y_0'$ are algebraically independent over the field of rational functions; and then the latter assertion, obviously, also holds for $K_{\lambda,\mu}(z)$ and $K'_{\lambda,\mu}(z)$. Applying Theorem 1 from ([7]), we obtain the assertion of the theorem.

Consider the functions

$$A_{\lambda,\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\nu+1)\cdots(\nu+n)}{(\lambda+1)\cdots(\lambda+n)(\mu+1)\cdots(\mu+n)} z^n,$$

which are solutions of the differential equation

$$y'' + \left(\frac{\lambda + \mu + 1}{z} - 1\right) y' + \frac{\lambda\mu - (\nu+1)z}{z^2} y = \frac{\lambda\mu}{z^2}.$$

**Theorem 2.** If $\lambda$, $\mu$, $\nu$ are rational numbers; $\lambda, \mu, \nu \neq -1, -2, \dots$; $\nu - \lambda \neq 0, 1, 2, \dots$; $\nu - \mu \neq 0, 1, 2, \dots$; and $\alpha \neq 0$ is any algebraic number, then the numbers $A_{\lambda,\mu,\nu}(\alpha)$ and $A'_{\lambda,\mu,\nu}(\alpha)$ are algebraically independent.

**Lemma 4.** If $\lambda$ and $\nu$ are complex numbers, $\nu \neq 0, \pm1, \pm2, \dots$; $\nu - \lambda \neq 0, \pm1, \pm2, \dots$; and $y \neq 0$ is any solution of the differential equation (2), then

$y$ satisfies no algebraic differential equation of the first order with polynomial coefficients.

The proof of Lemma 4 is analogous to the proof of Lemma 2 of paper ([3]), except that instead of the function $A_{\lambda,\mu}(z)$ one must consider an arbitrary solution $y \neq 0$ of equation (2) and first prove that it is not an algebraic function.

**Proof of Theorem 2.** Considering the function $y_0 = z^\mu A_{\lambda,\mu,\nu}(z)$, we see that it is a solution of the equation

$$y'' + \left(\frac{\lambda - \mu + 1}{z} - 1\right) y' - \frac{\nu - \mu + 1}{z} y = \lambda \mu z^{\mu - 2} \tag{16}$$

and, for $\nu \neq -1, -2, ...$, is not an algebraic function. Therefore the proof of the theorem for the case $\nu - \lambda \neq 0, 1, 2, ...$; $\nu - \mu \neq 0, 1, 2, ...$ proceeds as a literal repetition of the proof of Theorem 1, but with the use of equation (16) and Lemma 4. The cases when $\nu - \lambda \neq -1, -2, ...$ or $\nu - \mu \neq -1, -2, ...$ are considered analogously to the proof of Lemma 3 of paper ([3]), using Lemma 6 of paper ([8]).

With the aid of the arguments by which Lemma 1 and Theorem 1 were proved, as well as Lemma 9 of paper ([1]), one proves

**Theorem 3.** If $\lambda$, $\mu$, $\lambda_1$ are rational numbers distinct from negative integers, $\lambda_1$ and $\lambda - \mu$ are not equal to half an odd number, the numbers $\lambda - \mu \pm \lambda_1$ are not integers, and $\alpha \neq 0$ is any algebraic number, then the numbers $K_{\lambda_1}(\alpha)$, $K'_{\lambda_1}(\alpha)$, $K_{\lambda,\mu}(\alpha)$, $K'_{\lambda,\mu}(\alpha)$ are algebraically dependent.

Moscow State University
named after M. V. Lomonosov

Received
15 X 1965

**REFERENCES**

1. C. Siegel, Abh. Preuss. Acad. Wiss., No. 1 (1929-1930).
2. C. Siegel, *Transcendental Numbers*, Princeton, 1949.
3. V. A. Oleinikov, Vestn. Mosk. Univ., Math., Mech., No. 6 (1962).
4. A. B. Shidlovskii, DAN, 96, No. 4 (1954).
5. A. B. Shidlovskii, Scientific Notes of Moscow State University, Ser. Math., 9, issue 186 (1959).
6. A. B. Shidlovskii, DAN, 100, No. 2 (1955).

7. A. B. Shidlovskii, Izv. Acad. Sci. USSR, Ser. Math., 23, No. 1 (1959).

8. A. B. Shidlovskii, Proceedings of the Moscow Mathematical Society, 8, 283 (1959).

*Note: Figure translations are in progress. See original paper for figures.*