ANALYSIS OF THE DIRAC EQUATION IN RIEMANNIAN SPACE

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Abstract

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ANALYSIS OF THE DIRAC EQUATION IN RIEMANNIAN SPACE

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In Riemannian space, for spinors the covariant derivative is defined in the form

$$(\ )_{;\mu} = \nabla_\mu = \partial_\mu - A^a_\mu I_a, \tag{1}$$

where I_a is an infinitesimal operator that defines the algebra of the group corresponding to the given field A^a_μ.

This definition makes it possible to write the momentum operator, applying it to spinor functions, in the form

$$-\hat{p}_\mu = i\hbar\nabla_\mu = i\hbar(\partial_\mu - A^a_\mu I_a).$$

Let us now write the Dirac equation in covariant form (1)

$$L^\mu \partial_\mu \psi + k\psi = 0. \tag{2}$$

Here L^μ is the "spin" of the particle; $L^\mu = L^\nu e^\mu_\nu$; $k = mc/\hbar$. Equation (2) is covariant (invariant) with respect to the group of transformations

$$\psi' = \psi \exp[\varepsilon^a_{(x_i)} I_a].$$

In the case of the Lorentz group

$$\nabla_\mu = \partial_\mu - \tfrac{1}{2} M_{ik} \Delta_{\mu(i,k)}, \tag{3}$$

where M_{ik} is the generator of the Lorentz group, and

$$[M_{ik} M_{jl}] = f^{ps}_{ik;\,jl} M_{ps},$$

where $f^{ps}_{ik;\,jl}$ is the structural constant of the group.

The Ricci coefficients of parallel transport are expressed through the tetrads as follows:

$$\Delta_{\mu(i,k)} = e^{\tau}_{(i)} \nabla_\mu e_{\tau(k)}, \quad \text{where } \nabla_\mu e_{\tau(i)} = \partial_\mu e_{\tau(i)} - \Gamma^{\lambda}_{\mu\tau} e_{\lambda(i)}.$$

In this case the parallel transport of a vector ξ_i, if it is written in the orthonormal frame, will have the form

$$\delta\xi_i = \Delta_{\mu(i,k)} \xi_k \delta n^\mu.$$

The transport of any arbitrary quantity ψ_j will have the form

$$\delta\psi^j = \tfrac{1}{2} \Delta_{\mu(i,k)} M^{js}_{ik} \psi_s \delta n^\mu.$$

In the case of the spinor representation of the Lorentz group

$$M_{ik} = \tfrac{1}{2}[\gamma_i \gamma_k] = s_{ik}.$$

In this case the Dirac equation in curvilinear coordinates takes the form (^{1})

$$\gamma^i e^\mu_i \left(\partial_\mu - \frac{1}{4} [\gamma_i \gamma_k] \Delta_{\mu(i,k)} \right) \psi + k\psi = 0, \tag{4}$$

where $e^\mu_i e_{\mu,k} = \delta_{ik}, \quad e^\mu_i e^\nu_i = g^{\mu\nu}, \quad \tfrac{1}{2}[\gamma_i \gamma_k] = s_{ik} = M_{ik}$.

For a more explicit allowance for the gravitational field, let us square the Dirac equation, analogously to how this is done to allow for the electromagnetic field (^{2}):

$$\gamma^\mu \gamma^\nu (\partial_\mu - \tfrac{1}{2} s_{ik} \Delta_{\mu(i,k)})(\partial_\nu - \tfrac{1}{2} s_{jl} \Delta_{\nu(j,l)}) \psi =$$

$$= (g^{\nu\mu} - s^{\nu\mu}) \nabla_\mu \nabla_\nu \psi = g^{\nu\mu} \nabla_\nu \nabla_\mu \psi - \tfrac{1}{2} s^{\mu\nu} [\nabla_\mu \nabla_\nu] \psi = k^2 \psi. \tag{5}$$

Since

$$[\gamma^\mu \gamma^\nu]_+ = 2g^{\mu\nu}, \qquad s^{\mu\nu} = s^{ik} e^\mu_i e^\nu_k, \qquad [\nabla^\mu \nabla^\nu] = M_{ik} R_{\mu\nu(i,k)},$$

we shall have

$$g^{\mu\nu} [(\partial_\mu - \tfrac{1}{2} s_{ik} \nabla_{\mu(i,k)})(\partial_\nu - \tfrac{1}{2} s_{jl} \nabla_{\nu(j,l)})] \psi -$$

$$-\tfrac{1}{4}e^{\mu}_{p}e^{\nu}_{p}\gamma^{\gamma}\gamma^{p}\gamma^{j}\gamma^{l}R_{\mu\nu(j,l)}\psi = k^{2}\psi, \tag{6}$$

where $R_{\mu\nu(j,l)}$ is the curvature tensor in frames in orthogonal coordinates; it is expressed in terms of the general Ricci curvature tensor $R^{\tau}_{\mu\nu\lambda}$ in the following way:

$$R_{\mu\nu(j,l)} = e^{\lambda}_{j}e^{l}_{\tau}R^{\tau}_{\mu\nu\lambda}. \tag{7}$$

Since it is sufficient for us to make only an estimate of the influence of the gravitational field on the spin of the particle (and, moreover, a very small one, without being interested in still smaller details), we shall write

$$\tfrac{1}{4}e^{\mu}_{p}e^{\nu}_{p}\gamma^{\gamma}\gamma^{p}\gamma^{j}\gamma^{l}R_{\mu\nu(j,l)} = A^{ik}R_{ik} = R^{*},$$

where A^{ik} is some dimensionless tensor; R^{*} is a quantity proportional to the scalar curvature $R^{*} = \alpha R$.

Let us now rewrite equation (6) in the form

$$(g^{\mu\nu}\hat{p}_{\mu}\hat{p}_{\nu} + m^{2}c^{2} + \hbar^{2}R^{*})\psi = 0. \tag{8}$$

It is obvious that

$$\hat{p}_{\mu} = -i\hbar(\partial_{\mu} - \tfrac{1}{2}s^{ik}\Delta_{\mu(i,k)}). \tag{9}$$

We shall now clarify the meaning of the correction $\hbar^{2}R^{*} = Am^{2}c^{2}$, where A is some dimensionless scalar, and estimate the value of A. In the case of interaction with its own field we shall have:

$$R^{*} = -\alpha\chi T = 8\pi G\alpha\rho/c^{2},$$

where ρ is the density of matter producing the gravitational field; hence

$$A = \frac{\alpha\hbar^{2}8\pi G\rho}{m^{2}c^{4}} = \frac{\hbar}{r_0^{2}}\,\frac{\hbar G}{c^{3}}\,\frac{8\pi\alpha\rho r_0^{3}}{m^{2}cr_0}, \tag{10}$$

where r_0 is the characteristic radius of elementary particles, $r_0 \simeq 10^{-13}$ cm. The quantity $\hbar G/c^{3} = r_{\Phi}^{2}$, where $r_{\Phi} \simeq 10^{-33}$ cm is the characteristic magnitude of the metric fluctuation of the gravitational field. Let us denote the quantity

$$\hbar\,\frac{\hbar G}{r_0^{2}c^{3}} = \hbar\left(\frac{r_{\Phi}}{r_0}\right)^{2} = \hbar_{g} \tag{11}$$

and call $\hbar_g$ the "Planck constant" for gravitational interactions (^{3,4}).

Thus, $\hbar_g = \hbar\sigma_g/\sigma_p$, where σ_g and σ_p are the cross sections for gravitational and strong interactions, respectively. Further, since ${}^4/{}_3\pi\rho r_0^3 = m_p = m$,

then we shall have

$$A = 6\alpha\hbar_g/mcr_0 \simeq \hbar_g/\hbar = \sigma_g/\sigma_p = (r_\Phi/r_0)^2 \simeq 10^{-40}.$$

In the more general case, if the "radius" of curvature $r^* \neq r_0$, $A = r_0 r_\Phi^2/r^{*3}$. If the effective "radius" of curvature is taken to be $r^* = r_\Phi = 10^{-33}$ cm (5), then $A = r_0/r_\Phi = 10^{20}$.

Since $m^2c^2 + \hbar^2 R^* = m^2c^2(1+A) = m_0^2c^2$, where m_0 is the mass "reduced" in the gravitational field, then

$$m_0 = m\sqrt{1+A} = m\sqrt{1+\sigma_g/\sigma_p}. \tag{12}$$

In an external field (in the field of the Metagalaxy) $R^* = 12a/a^2$, where a is the radius of curvature of the Metagalaxy; $A = 12\alpha M_0 r_0^2\hbar_g/m^2ca^3$; since $M_0/a^2 \simeq m/r_0^2$, we have

$$A = \frac{12\alpha\hbar_g}{mac} \simeq \frac{\hbar_g}{\hbar}\frac{r_0}{a} = \left(\frac{\hbar_g}{\hbar}\right)^2 \simeq 10^{-80}.$$

Here $M_0 \simeq 10^{80}m$ is the mass of the Metagalaxy. Since the quantity a varies with time, $a \sim t$, it follows that $R^* \sim t^{-2}$, and the quantity A also decreases with time. Thus, the interaction of particles, or, more precisely, the interaction of particle spin, changes (decreases) with time.

The expression $m^2c^2 + \hbar^2 R^*$ in the case of an external field can also be written in the form $\hbar^2(1/r_0^2 + \text{const}/a^2)$; here it is evident that the quantities $\hbar c/r_0 = E_p$, $\hbar c/a \simeq E_g = E_p r_0/a \simeq E_p \cdot 10^{-40}$ characterize, respectively, the energy of the nucleon and of the graviton.

In investigating such small energy corrections and distances of order $r_0 \cdot 10^{-20} = 10^{-33}$ cm, there is, of course, no complete certainty that the Dirac equation is applicable to such situations. But the fact that these small corrections lead to effects of the gravitational field that are correct in order of magnitude indicates the applicability of the Dirac equation in a gravitational field.

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