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FACTORIZATION OF THE S-MATRIX OF A MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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In the present paper certain formulas of scattering theory, known for the one-dimensional Schrödinger equation,

$$Hu \equiv -u'' + v(x)u = k^2 u, \quad k > 0, \tag{1}$$

on the whole axis $-\infty < x < \infty$, are generalized to the multidimensional case. If the real potential $v(x)$ decreases sufficiently rapidly as $|x| \to \infty$, then the following formulation of the scattering problem is meaningful: find solutions $\psi_1(x,k)$ and $\psi_2(x,k)$ of equation (1) having the asymptotics

$$\begin{array}{lll} \psi_1(x,k) \sim e^{ikx} + s_{12}(k)e^{-ikx}, & \psi_2(x,k) \sim s_{22}(k)e^{-ikx}; & x \to -\infty; \\ \psi_1(x,k) \sim s_{11}(k)e^{ikx}; & \psi_2(x,k) \sim e^{-ikx} + s_{21}(k)e^{ikx}; & x \to \infty. \end{array} \tag{2}$$

The matrix $S(k)$ with elements $s_{ij}(k)$, $i,j = 1,2$, is called the S-matrix of the problem under consideration. From general considerations of scattering theory it follows that it is unitary and symmetric,

$$\overline{s}_{11}s_{21} + \overline{s}_{12}s_{22} = 0; \quad |s_{11}|^2 + |s_{12}|^2 = 1; \quad s_{11} = s_{22}. \tag{3}$$

A rigorous proof of the existence of a solution of this problem is given in (^{1}). It is shown there also that the matrix element $s_{11}(k)$ has an analytic continuation into the upper half-plane of the variable k, has there simple poles at the points $k = i\chi_l$, where χ_l^2, $l = 1, \dots, m$, are the discrete eigenvalues of the operator H_1, and does not vanish. With the aid of (3) one can verify that the relations

$$S(k) = M_1^{(-)}(k)\,[M_1^{(+)}(k)]^{-1} = M_2^{(-)}(k)\,[M_2^{(+)}]^{-1}, \tag{4}$$

hold, where the triangular matrices $M_1^{(+)}$, $M_2^{(+)}$, $M_1^{(-)}$ and $M_2^{(-)}$ have respectively the form

$$\begin{pmatrix} m_{11} & m_{12} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ m_{21} & m_{22} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \overline{m}_{12} & \overline{m}_{11} \end{pmatrix}, \quad \begin{pmatrix} \overline{m}_{22} & \overline{m}_{21} \\ 0 & 1 \end{pmatrix};$$

$$m_{11} = m_{22} = s_{11}^{-1}; \quad m_{12} = -s_{12}s_{11}^{-1}; \quad m_{21} = -s_{21}s_{22}^{-1}.$$

The determinant of the matrices $M_i^{(+)}$, $i = 1, 2$, is the function $\Delta(k) = s_{11}(k)^{-1}$. As follows from what was said above, this is an analytic function in the upper half-plane, having zeros at the points $i\chi_l$, $l = 1, \dots, m$. Factorization (4) was first mentioned in the work of Kay and Moses (^{2}), devoted to the inverse problem of scattering theory, i.e. the problem of reconstructing the potential $v(x)$ from the S-matrix.

It is known that equation (1) has two solutions $f_1(x,k)$ and $f_2(x,k)$, defined by the asymptotics as $x \to \infty$ and $x \to \infty$ respectively,

$$\exp\{-ikx\}f_1(x,k) \to 1, \quad x \to \infty; \qquad \exp\{ikx\}f_2(x,k) \to 1, \quad x \to -\infty.$$

Through the elements of the matrices $M_i^{(\pm)}(k)$ one expresses the asymptotics of these solutions, for example as $x \to -\infty$

$$f_1(x,k) \sim \exp\{ikx\}m_{11} - \exp\{-ikx\}m_{12}.$$

Thus, the factorization (4) is closely connected with the existence of the solutions $f_1(x,k)$ and $f_2(x,k)$.

In a recent paper (3) the author proposed a multidimensional analogue of these solutions. It is natural to examine what the corresponding factorization of the S-matrix looks like and to what extent the properties of the operators participating in it are a generalization of the properties of the matrices $M_i^{(\pm)}$.

Consider the n-dimensional Schrödinger operator

$$H_n u = -\Delta u + v(x)u; \qquad x = (x_1, \dots, x_n) \in E_n.$$

The outgoing $(+)$ and incoming $(-)$ solutions of the scattering problem are defined as solutions of the equation $H_n\psi^{(\pm)} = k^2\psi^{(\pm)}$, having the following asymptotics as $|x| \to \infty$:

$$\psi^{(\pm)}(x,k) - \exp\{i(k,x)\} \sim \exp\{\pm i|k||x|\}|x|^{-1}f^{(\pm)}(k,\omega);$$

$$k = (k_1, \dots, k_n); \qquad (k, x) = k_1 x_1 + \dots + k_{nx} n;$$

$$k^2 = (k, k); \qquad |k| = (k^2)^{1/2}; \qquad \omega = x|x|^{-1}.$$

The function $f^{(\pm)}(k, \omega)$ is called the **scattering amplitude**. The existence of a solution of the formulated problem was proved by A. Ya. Povzner (4) by means of the integral equation

$$\psi^{(\pm)}(x, k) = \exp\{i(k, x)\} - \int G^{(\pm)}(x - y, k^2) v(y) \psi^{(\pm)}(y, k)\, dy, \tag{5}$$

where $G^{(\pm)}(x, k^2)$ are the Green' s functions, having the integral representations

$$G^{(\pm)}(x, k^2) = (2\pi)^{-n} \int \exp\{i(m, x)\}(m^2 - k^2 \mp i0)^{-1}\, dm. \tag{6}$$

The **scattering operator** is the integral operator S with kernel

$$S(k, l) = \delta(k - l) - i(2\pi)^{1-n} \delta(k^2 - l^2) f(k, l), \tag{7}$$

where the kernel $f(k, l)$ is expressed through $\psi^{(+)}(x, l)$ by the formula

$$f(k, l) = \int \exp\{-i(k, x)\} v(x) \psi^{(+)}(x, l)\, dx.$$

The operator S is unitary in $L_2(E_n)$ and satisfies the symmetry condition, which in terms of the kernel $f(k, l)$ has the form

$$f(k, l) = f(-l, -k), \qquad k^2 = l^2.$$

The solutions $\psi^{(+)}(x, l)$ can be represented in the form of the following linear combination of the solutions $\psi^{(-)}(x, k)$:

$$\psi^{(+)}(x, l) = \int \psi^{(-)}(x, k) S(k, l)\, dk. \tag{8}$$

Let us note that for $|k| = |l|$ the equality $f(k, |k|\omega) = 4\pi f^{(+)}(k, \omega)$ holds, so that the scattering operator is determined by the scattering amplitude. The presence of $\delta(k^2 - l^2)$ in the definition (7) of the operator shows that this operator generates on the unit sphere a family of integral operators $S(k^2)$. This family is an analogue of the S-matrix of the one-dimensional operator H_1.

We now introduce a multidimensional analogue of the solutions $f(x,k)$. In (3) the Green' s function

$$G_q(x,k) = (2\pi)^{-n} \int \exp\{i(m,x)\}[(m+iq)^2 - (k+iq)^2]^{-1}\, dm \tag{9}$$

was considered.

Let $\gamma = q|q|^{-1}$. In (3) it was shown that $G_q(x,k)$ is an analytic function of the variable $s = (k,\gamma) + i|q|$. Here we shall consider

the limiting value of this function for $\operatorname{Im} s = |q| = 0$, which depends on γ. We shall denote it by $G_\gamma(x,k)$. The kernel $G_\gamma(x,k)$ is bounded for $|x| \neq 0$. Define the solutions $\varphi_\gamma(x,k)$ of the equation $H_n\varphi = k^2\varphi$ as solutions of the integral equation

$$\varphi_\gamma(x,k) = \exp\{i(k,x)\} - \int G_\gamma(x-y,k)v(y)\varphi_\gamma(y,k)\, dy. \tag{10}$$

It can be shown that if the potential $v(x)$ is continuous and decreases sufficiently rapidly as $|x| \to \infty$, then this equation is Fredholm. Unfortunately, we have not succeeded in proving the absence of nontrivial solutions of the corresponding homogeneous equation, so that the existence of the solutions $\varphi_\gamma(x,k)$ has not been proved. In what follows we shall assume that equation (10) is uniquely solvable for all k and γ. This, in any case, holds for sufficiently small $v(x)$.

From comparison of the integral representations (9) and (6) it follows that

$$G_\gamma(x,k) = G^{(\pm)}(x,k^2) \mp i(2\pi)^{1-n} \int \exp\{i(m,x)\}\delta(m^2-k^2)\theta[\pm(m-k,\gamma)]\, dm.$$

Here $\theta(a) = 1,\ a > 0;\ \theta(a) = 0,\ a < 0$. With the aid of these relations and the integral equations (5) and (10), one can show that the solutions $\varphi_\gamma(x,l)$ are the following linear combinations of the solutions $\psi^{(\pm)}(x,k)$:

$$\varphi_\gamma(x,l) = \int \psi^{(\pm)}(x,k)Q_\gamma^{(\pm)}(k,l)\, dk, \tag{11}$$

where the kernels $Q_\gamma^{(\pm)}(k,l)$ of the integral operators $Q_\gamma^{(\pm)}$ are expressed in terms of the kernel

$$h_\gamma(k,l) = \int \exp\{-i(k,x)\}v(x)\varphi_\gamma(x,l)\, dx$$

as follows:

$$Q_\gamma^{(\pm)}(k,l) = \delta(k-l) \pm i(2\pi)^{1-n} h_\gamma(k,l)\theta[\pm(k-l,\gamma)]\delta(k^2-l^2). \tag{12}$$

The operators $Q_\gamma^{(\pm)}$ generate, on the unit sphere, a family of integral operators $Q_\gamma^{(\pm)}(k^2)$.

Let us rewrite relations (8) and (11) briefly in the form

$$\psi^{(+)} = \psi^{(-)} S; \qquad \varphi_\gamma = \psi^{(\pm)} Q_\gamma^{(\pm)},$$

whence the factorization of the operators S and $S(k^2)$ is visible:

$$S = Q_\gamma^{(-)}[Q_\gamma^{(+)}]^{-1}; \qquad S(k^2) = Q_\gamma^{(-)}(k^2)[Q_\gamma^{(+)}(k^2)]^{-1}. \tag{13}$$

We regard (13) as a multidimensional analogue of formulas (4).

Let us compare the properties of the operators $Q_\gamma^{(\pm)}(k^2)$ and the matrices $M_i^{(\pm)}(k)$. The analogue of the triangularity of the matrices $M_i^{(\pm)}(k)$ is the Volterra property of the operators $Q_\gamma^{(\pm)}(k^2)$, which follows explicitly from the presence of the θ-function in expressions (12). Further, on the diagonals of the matrices $M^{(+)}$ there stand functions that have an analytic continuation into the complex plane. The operators $Q^{(+)}(k^2)$ possess an analogous property. Indeed, the solutions $\varphi_\gamma(x,k)$ have an analytic continuation into the upper half-plane with respect to the variable $s=(k,\gamma)$. It is precisely this analytic continuation $\varphi_\gamma(x,k_\perp,s)$, $k_\perp = k-(k,\gamma)\gamma$, that was considered in (3), where it was shown that $\exp\{-is(x,\gamma)\}\varphi_\gamma(x,k_\perp,s)$ is bounded for all x. We see that the amplitude $h_\gamma(k,l)$, for $(k,\gamma)=(l,\gamma)=s$, also has an analytic continuation with respect to s. From (12) it is clear that precisely this amplitude stands on the diagonal of the Volterra operators $Q_\gamma^{(+)}(k^2)$.

Owing to the Volterra property of the operators $Q^{(+)}(k^2)$, their determinants are expressed explicitly in terms of the kernel, and the formula holds

$$\Delta_\gamma(\lambda) = \det Q_\gamma^{(+)}(\lambda) = \exp\left\{ i(2\pi)^{1-n} \int h_\gamma(k,k)\delta(k^2-\lambda)\,dk \right\},$$

whence one can show that these determinants have an analytic continuation in the plane of the variable λ with a cut along the positive part of the real axis. Unfortunately, as was already noted in (3), we do not know the location of the singular points of the solution $\varphi_\gamma(x,k_\perp,s)$, and hence of the analytic continuation of the amplitude $h_\gamma(k,l)$ and of the determinant $\Delta_\gamma(\lambda)$ itself. There are grounds to assert that $\Delta_\gamma(\lambda)$ is analytic on the entire plane with a cut and has zeros at the discrete eigenvalues of the operator H_n; however, a rigorous proof of this assertion is lacking.

The unitarity of the operator S and the Volterra character of the operators $Q_\gamma^{(\pm)}$ are compatible only under the condition that $[Q_\gamma^{(\pm)}]^{-1} = [Q_\gamma^{(\pm)}]^*$. This relation indeed holds, as can be verified by means of the integral equation (10). The corresponding relation in the one-dimensional case looks somewhat different.

In the one-dimensional case the matrices $M_i^{(\pm)}(k)$ are expressed explicitly in terms of the matrix elements of the S-matrix. In the case $n > 1$ there are no such explicit expressions; however, in principle, from a given operator S one can reconstruct the operators $Q_\gamma^{(\pm)}$. Indeed, rewriting (13) in the form $SQ_\gamma^{(+)} = Q_\gamma^{(-)}$ and substituting here (7) and (12), we obtain the relation

$$h_\gamma(k,l) = f(k,l) + i(2\pi)^{1-n} \int f(k,m)\,\theta[(m-l,\gamma)]\,h_\gamma(m,l)\,\delta(m^2-k^2)\,dm,$$

valid for $k^2 = l^2$, which can be regarded as an equation for determining the kernel $h_\gamma(k,l)$ in terms of $f(k,l)$.

We may conclude that the analogy between formulas (13) and (4) extends quite far.

The results presented in (3) and in the present work were obtained by the author in the course of seeking a formalism suitable for solving the inverse scattering problem in the multidimensional case. The close analogy between the solutions $f(x,k)$ and $\varphi_\gamma(x,k)$ that we have traced makes it possible to hope that the latter, like the former, may serve as a basis for solving this problem. Indeed, if one assumes that the solutions $\varphi_\gamma(x,k_\perp,s)$ have no singularities for all s, then one can obtain a linear integral equation of Gelfand–Levitan type, relating the kernel $A_\gamma(x,k_\perp,t)$ of the transformation operator introduced in (3) to the kernel of the operator $W_\gamma = Q_\gamma^{(+)*}Q_\gamma^{(+)}$, and hence to the scattering operator. It is not known, however, how to write such an equation in the general case, when $\varphi_\gamma(k,k_\perp,s)$ has singularities. The difficulty consists in the fact that we do not know the detailed characteristics of these singularities. Thus, the main unsolved problem in the subject connected with the solutions $\varphi_\gamma(x,k)$, as we have already noted more than once, is the investigation of the solutions of the homogeneous equation corresponding to (10).

In conclusion, we note that our generalization of relation (4) differs from that proposed by Moses in (5), and the formalism for solving the inverse scattering problem developed by Kay and Moses in (6) appears to us to be erroneous.

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