AMORPHOUS FUNCTIONALS, THEIR STRUCTURE AND THEIR APPLICATIONS

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Abstract

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AMORPHOUS FUNCTIONALS, THEIR STRUCTURE AND THEIR APPLICATIONS

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Absolutely monotone segments-functionals defined on the set of polynomials of degree n split into two essentially different classes (^{1}).

I. A **nodal functional** $(a_k)_0^n$ has an extremal polynomial $Q_n(x)$, where $n \geq 1$, $\max_{[0,1]} |Q_n| = 1$, and, correspondingly, a unique best extension to the set of continuous functions with norm $N = a_0$. The structure of such a functional has the form

$$a_k = \sum \delta_i \sigma_i^k \quad (k = 0, 1, \dots, n), \tag{1}$$

where (σ_i) are the nodes of the functional, $0 \leq \sigma_i \leq 1$; $Q_n(\sigma_i) = +1$ for $\delta_i > 0$. The decomposition (1) for $(a_k)_0^n$ is unique, and the nodes (σ_i) are its true nodes. The number of nodes S is strictly majorized (Theorem 1). The simplest nodal functional is $A, Aq, Aq^2, \dots, Aq^n$ $(n \geq 2)$, $A > 0$; $0 \leq q \leq 1$; we shall call it a **quota**.

II. An **amorphous functional** $(a_k)_0^n$ has no extremal polynomial of degree $n \geq 1$; its norm a_0 is attained only on $Q(x) \equiv 1$; its best extension a_{n+1} to polynomials of degree $(n+1)$ may fill a certain interval $\alpha'_{n+1} < \alpha_{n+1} < \alpha''_{n+1}$. Such a segment has no definite nodal structure. If $\alpha_{n+1} = \alpha'_{n+1}$ or $\alpha_{n+1} = \alpha''_{n+1}$, then the segment $(a_k)_0^{n+1}$ becomes nodal.

Theorem 1. *Let the segment $(a_k)_0^n$ be decomposed into the sum of quotas*

$$a_k = \sum_{i=1}^{S} \delta_i \sigma_i^k$$

and let their number S satisfy the conditions: for even n,

$$S \geq \frac{n}{2} + 1$$

with the reservation: if $S = n/2+1$*, then the nodes do not simultaneously contain both* 0 *and* 1*; for odd* n*,* $S \geq (n+1)/2$ *with the reservation: if* $S = (n+1)/2$*, then all* $0 < \sigma_i < 1$ *(interior nodes). Then* $(a_k)_0^n$ *is amorphous.*

These assertions follow from the impossibility of constructing an extremal polynomial of degree $\leq n$ with such a number of nodes (^{1}).

The converse assertion is also true: an amorphous segment can be (and in infinitely many ways) decomposed into quotas with number S satisfying the conditions mentioned. The nodes of such quotas will be called **fictitious nodes of the amorphous segment**, and the decomposition will be called **fictitious**.

Such are, for example, the true decompositions of the segments: $\alpha_0, \dots, \alpha_n, \alpha'_{n+1}$ or $\alpha_0, \dots, \alpha_n, \alpha''_{n+1}$ or $\alpha_0, \dots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}$, etc. ($\alpha'_{n+1} < \alpha_{n+1} < \alpha''_{n+1}$). These decompositions are fictitious for $(a_k)_0^n$.

Corollary 1. *The trueness of a set of nodes* $(\sigma_i)_1^S$ *for a nodal segment* $(a_k)_0^n$ *is determined only by the number* S*, and not by the arrangement of the points* (σ_i) *on* $(0,1)$*.*

Let us recall that the nodes and their weights (δ_k) are completely and uniquely determined (computed) from the parameters (a_k).

Corollary 2. Any fictitious decomposition of an amorphous interval

$$a_k = \sum_1^s \delta_i \sigma_i^k \quad (k = 0, 1, \dots, n)$$

becomes a true one under a sufficiently far continuation "along these nodes," i.e., if one sets

$$\alpha_{n+1} = \sum_1^s \delta_i \sigma_i^{n+1}, \quad \alpha_{n+2} = \sum_i^s \delta_i \sigma_i^{n+2}, \quad \text{etc.}$$

Theorem 2. From every amorphous $(a_k)_0^n$ one can subtract (without violating absolute monotonicity) a quota of any node q ($0 \leq q \leq 1$), i.e., the interval $(a_k - \delta q^k)_0^n$ is absolutely monotone for sufficiently small $\delta > 0$.

We give the proof (constructive!). Let us first note that from a nodal interval one can subtract only the quotas of its true nodes, which is immediately proved by contradiction.

1) $q = 1$. The interval $a_0, a_1, \dots, a_n, \alpha''_{n+1}$ is nodal and contains the node 1.
2) $q = 0$. For even n the same interval contains the node 0; for odd n we take $a_0, a_1, \dots, a_n, C_{n+1}$, and it contains the node 0.
3) $0 < q < 1$, and suppose that among the true nodes of the two intervals

$$a_0, \dots, a_n,\ \alpha''_{n+1} \tag{2}$$

$$a_0, \dots, a_n,\ \alpha'_{n+1} \tag{3}$$

there is no node q (for if it is contained, then the problem is solved).

Recall that an interval (not absolutely monotone) of the form $(\mu_k)_0^n = 0_0, 0_1, \dots, 0_n, \varepsilon_{n+1}$, where $\varepsilon > 0$, can be represented in the form $\mu_k = \sum_1^p \Delta_i \lambda_i^k$ $(k = 0, 1, \dots, n)$, where $0 \le \lambda_i \le 1$, only under the condition $p \ge n+2$, and, if $p = n+2$, then all Δ_i have alternating signs: $\Delta_{n+2} > 0$; $\Delta_{n+1} < 0, \dots$

For definiteness assume that n is even. The nodes of the interval (3), in number $n/2+1$, we take to be $\lambda_{n+1}, \lambda_{n-1}, \dots$, and as $\lambda_{n+2}, \lambda_n, \dots$ we take any nodes alternating with them, including the number q. Then for sufficiently small ε the interval $a_0, \dots, a_n, \alpha'_{n+1} + \varepsilon$ has a fictitious decomposition into $n+2$ quotas, and $(a_k)_0^n$ contains the node q.

Corollary 1. There exists a maximal weight δ such that the interval

$$(\beta_k)_0^n = (a_k - \delta q^k)_0^n \tag{4}$$

becomes nodal, while not containing the node q.

Corollary 2. A more detailed investigation of the results of subtracting a single quota from an amorphous $(a_k)_0^n$ leads to the following conclusions: the degree of the interval $(\beta_k)_0^n$ in (4) can decrease by no more than two units, i.e., the interval $(\beta_k)_0^{n-3}$ is amorphous in all cases.

Corollary 3. The true nodes of the intervals (2) and (3) alternate with one another.

Indeed, let us first establish that (2) and (3) have no common nodes: the interval (2) has $n/2+1$ or $(n+1)/2+1$ nodes, while (3) has $n/2+1$ or $(n+1)/2$ (depending on whether n is even or odd). The combined number of nodes is $n+2$, but if any coincide it is $\le n+1$, which is impossible, since these are the (fictitious) nodes of the interval $0_0, 0_1, \dots, 0_n, \varepsilon$, where $\varepsilon = a_{n+1} - a'_{n+1} (> 0)$, and their number cannot be less than $n+2$. Hence the alternation of the signs of their weights immediately implies, and consequently the alternation of the nodes themselves.

Appendix. First of all recall that a not absolutely monotone interval $(\mu_k)_0^n$ can have an infinite set of nodal decompositions, but among them only one is true, and the others are fictitious.

A true decomposition

$$\mu_k = \sum_1^{S_1} \delta'_i \sigma'^k_i - \sum_1^{S_2} \delta''_i \sigma''^k_i \quad (k = 0, 1, \dots, n)$$

is one for which $0 \leqslant \sigma_i', \sigma_i'' \leqslant 1$; $\delta_i' > 0$; $\delta_i'' > 0$, and there exists a polynomial $Q_n(x)$ such that $\max_{[0,1]} |Q_n(x)| = 1$, $Q_n(\sigma_i') = +1$, $Q_n(\sigma_i'') = -1$. This polynomial is extremal for the segment-functional $F_n = (\mu_i)_0^n$, i.e.

$$N = F_n(Q_n) = \sum_1^{S_1} \delta_i' + \sum_1^{S_2} \delta_i''.$$

Theorem 3. Let $(\mu_k)_0^n$ have a double nodal decomposition (one of them may be true),

$$(\mu_k =) \sum_1^{S_1} \delta_i' \sigma_i'^k - \sum_1^{S_2} \delta_i'' \sigma_i''^k = \sum_1^{\bar{S}_1} \bar{\delta}_i' \bar{\sigma}_i'^k - \sum_1^{\bar{S}_2} \bar{\delta}_i'' \bar{\sigma}_i''^k \quad (k = 0, 1, \dots, n). \tag{5}$$

If all nodes do not leave $(0, 1)$, then either the two decompositions are identical, or

$$\bar{S}_1 + S_2 \geq \begin{cases} n/2 + 1, \\ (n+1)/2, \end{cases} \tag{6}$$

under the qualifications of Theorem 1, and the same inequalities hold for $S_1 + \bar{S}_2$.

Indeed, from (5) it follows that

$$\sum_1^{S_1} + \sum_1^{\bar{S}_2} = \sum_1^{\bar{S}_1} + \sum_1^{S_2} \quad (k = 0, 1, \dots, n). \tag{7}$$

If nodal segments occur on both sides of this equality, then, by the uniqueness of the decomposition, they must coincide identically, i.e.

$$\sum_1^{S_1} \equiv \sum_1^{\bar{S}_1} \quad \text{and} \quad \sum_1^{S_2} \equiv \sum_1^{\bar{S}_2}.$$

If, however, both segments in (7) are amorphous, then, according to Theorem 1, the inequalities (6) are valid.

Remark 1. If one sets $S_1 + \bar{S}_2 = S$ and $\bar{S}_1 + S_2 = \bar{S}$, then, according to a known theorem ((^{1}), p. 66), one has $S + \bar{S} \geq n + 2$; moreover, it is not required that the nodes lie in $(0, 1)$; they may even be complex.

Theorem 3, by restricting the position of the nodes to the segment $(0, 1)$, gives an estimate separately for each of the two mentioned paired summands entering into $(S + \bar{S})$.

Remark 2. For the number of true nodes of the segment $(\mu_k)_0^n$, evidently, the majorants

$$S_1, S_2 \leq \begin{cases} n/2+1, \\ (n+1)/2, \end{cases}$$

always hold; moreover, if $S_1 = n/2+1$, then $S_2 \leq n/2$, and conversely.

We give several examples of applications of Theorem 3.

Example 1. Let $(\mu_k)_0^n$ be any non-amorphous segment, and let

$$\mu_k = \sum_1^{S_1} \delta_i' \sigma_i'^k - \sum_1^{S_2} \delta_i'' \sigma_i''^k \quad (k = 0, 1, \dots, n) \tag{8}$$

be its true decomposition.

Consider the segment

$$\mu_0, \ \mu_1, \ \dots, \ \mu_n, \ \Theta_{n+1}, \tag{9}$$

and suppose that, for some $\Theta \neq \mu_{n+1}^*$ (μ_{n+1}^* is the unique best contin-

···its true decomposition is

$$\sum_1^{\bar{S}_1} \bar{\delta}_i' \bar{\sigma}_i'^{\,k} - \sum_1^{\bar{S}_2} \bar{\delta}_i'' \bar{\sigma}_i''^{\,k} \quad (k = 0, 1, \dots, n, n+1).$$

Then, for $k = 0, 1, \dots, n$, an equality of the form (7) and the inequalities (6) hold. Choosing as $(\mu_k)_0^n$ an absolutely monotone segment, we have $S_2 = 0$; consequently, $\bar{S}_1 \geq n/2+1$ or $\geq (n+1)/2$, i.e., taking into account Remark 2 and the qualifications of Theorem 1, we have:

$$1) \quad \bar{S}_1 = \frac{n}{2} + 1; \qquad 2) \quad \frac{n+1}{2} < \bar{S}_1 < \frac{n+1}{2} + 1.$$

Thus, if $(\mu_k)_0^n$ is absolutely monotone, then for any Θ an extremal polynomial $Q_{n+1}(x, \Theta)$ of the segment-functional (9) either has a maximum number of (+) nodes, i.e. is a primitive polynomial with internal deformation $W_n(x)$ (1), or $Q_{n+1}(x, \Theta) = T_{n+1}(\alpha x + \beta)$. Suppose, in addition, that $\mu_k = \delta q^k$; then $S_1 = 1$, $S_2 = 0$; in this case the extremal polynomial is $W_n(x)$ of passport $[n, n, 1]$.

Example 2. Consider two segment-functionals

$$\mu_0, \ \mu_1, \ \dots, \ \mu_n, \tag{10}$$

$$\mu_0,\ \mu_1\xi,\ \dots,\ \mu_n\xi^n, \qquad \xi > 0. \tag{11}$$

Put

$$\mu_k = \sum_1^{S_1} \delta_i' {\sigma_i'}^k - \sum_1^{S_2} \delta_i'' {\sigma_i''}^k, \tag{12}$$

$$\mu_k\xi^k = \sum_1^{\bar{S}_1} \bar{\delta}_i' {\bar{\sigma}_i'}^k - \sum_1^{\bar{S}_2} \bar{\delta}_i'' {\bar{\sigma}_i''}^k \tag{13}$$

as their true decompositions.

A. For $0 < \xi < 1$, (11) also has a fictitious decomposition

$$\mu_k\xi^k = \sum_1^{S_1} \delta_i' (\sigma_i'\xi)^k - \sum_1^{S_2} \delta_i'' (\sigma_i''\xi)^k \quad (k = 0, 1, \dots, n),$$

in which the nodes $(\sigma_i'\xi)$ and $(\sigma_i''\xi)$ remain in $(0, 1)$.

B. For $\xi > 1$, (10) also has a fictitious decomposition

$$\mu_k = \sum_1^{\bar{S}_1} \bar{\delta}_i' \left(\frac{\bar{\sigma}_i'}{\xi}\right)^k - \sum_1^{\bar{S}_2} \bar{\delta}_i'' \left(\frac{\bar{\sigma}_i''}{\xi}\right)^k \quad (k = 0, 1, \dots, n),$$

in which the nodes $(\bar{\sigma}_i'/\xi)$ and $(\bar{\sigma}_i''/\xi)$ do not go outside the segment $[0, 1]$.

Thus, on the basis of Theorem 3, the inequalities hold

$$\bar{S}_1 + S_2 \geq \begin{cases} n/2 + 1, \\ (n+1)/2, \end{cases} \qquad \bar{S}_2 + S_1 \geq \begin{cases} n/2 + 1, \\ (n+1)/2. \end{cases}$$

Example 3. Let the segment $(\mu_k)_0^n$ have, in addition to the true distribution

$$\mu_k = \sum_1^{S_1} \delta_i' {\sigma_i'}^k - \sum_1^{S_2} \delta_i'' {\sigma_i''}^k,$$

a fictitious two-node distribution

$$\mu_k = \Delta_1\lambda_1^k - \Delta_2\lambda_2^k \quad (k = 0, 1, \dots, n),$$

where $0 < \lambda_1 < \lambda_2 < 1$ and they are placed sufficiently close to be fictitious. Here $\bar{S}_2 = 1$ and $\bar{S}_1 = 1$. Restricting ourselves to even n, we obtain: $S_1 \geq n/2$, $S_2 \geq n/2$. Thus, the following combinations are possible: 1) $S_1 = n/2 + 1$, $S_2 = n/2$, or conversely. In these cases the functional is served by Chebyshev polynomials $\pm T_n(x) = \pm \cos n \arccos(2x - 1)$; 2) $S_1 = n/2$, $S_2 = n/2$. The total number of nodes is $S = n$, and, owing to the equality $S_1 = S_2$, the functional can be served only by Zolotarev polynomials.

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1. E. V. Voronovskaya, *The Method of Functionals and Its Applications*, 1963.
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Note: Figure translations are in progress. See original paper for figures.