# ON MIXED APPROXIMATIONS OF FUNCTIONS OF A COMPLEX VARIABLE IN OPPOSITE ANGLES BY MEANS OF ENTIRE FUNCTIONS

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## Abstract

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# ON MIXED APPROXIMATIONS OF FUNCTIONS OF A COMPLEX VARIABLE IN OPPOSITE ANGLES BY MEANS OF ENTIRE FUNCTIONS

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The present note is devoted to the application of the following principle in the theory of approximation of functions of a complex variable: the behavior of the best approximation of a function of a complex variable from a given class, in the metric of a given functional space in infinite domains bounded by straight lines, by means of entire analytic functions can be studied on the basis of the behavior of the same quantities along the boundary straight lines.

Denote by $D_1^{(\rho)}$ and $D_2^{(\rho)}$, respectively, the domains of the opposite angles

$$|\arg z| < \frac{\pi}{2}\left(1-\frac{1}{\rho}\right) \quad \text{and} \quad |\arg z+\pi| < \frac{\pi}{2}\left(1-\frac{1}{\rho}\right),$$

and set

$$\overline{D}^{(\rho)} = \overline{D}_1^{(\rho)} + \overline{D}_2^{(\rho)}, \quad D^{(\rho)} = D_1^{(\rho)} + D_2^{(\rho)},$$

where $\rho \geq 1$ is a given number.*

Further, denote by $\mathcal{L}_p^*[\overline{D}^{(\rho)}]$ the set of functions $f(z)$ for which

$$\int_{C(\varphi)} |f(z)|^p |dz| < M < +\infty \quad (p \geq 1)$$

for every $\varphi$ in the interval

$$|\varphi| \leq \frac{\pi}{2}\left(1-\frac{1}{\rho}\right),$$

where $C(\varphi)$ is the straight line with equation

$$z = te^{i\varphi} \left( -\infty < t < \infty, \ |\varphi| \leq \frac{\pi}{2}\left(1 - \frac{1}{\rho}\right) \right),$$

and $M$ does not depend on $\varphi$.

In particular, the set of functions from $\mathcal{L}_p^*[\overline{D}^{(\rho)}]$ analytic inside $D^{(\rho)}$ is denoted by $\mathcal{L}_p[\overline{D}^{(\rho)}]$. Obviously,

$$\mathcal{L}_p[D^{(\rho)}] \subset \mathcal{L}_p^*[\overline{D}^{(\rho)}].$$

Let us note that if, for a function $f(z)$ from the set $\mathcal{L}_p[\overline{D}^{(\rho)}]$, one defines the norm

$$\|f\|_{\mathcal{L}_p[\overline{D}^{(\rho)}]} = \sup_{|\varphi| < \frac{\pi}{2}\left(1 - \frac{1}{\rho}\right)} \left\{ \int_{C(\varphi)} |f(z)|^p |dz| \right\}^{1/p},$$

then the set $\mathcal{L}_p[\overline{D}^{(\rho)}]$ turns out to be a linear normed space.

Further, denote by $\mathcal{L}_p^{(\sigma)}[\overline{D}^{(\rho)}]$ the set of entire functions** of order $\rho$ $(1 \leq \rho < +\infty)$ and type $\sigma$, belonging to $\mathcal{L}_p^*[\overline{D}^{(\rho)}]$. We shall call the quantity

$$A_\sigma(f; \overline{D}^{(\rho)})_p = \inf_{g \in \mathcal{L}_p^{(\sigma)}[\overline{D}^{(\rho)}]} \|f(z) - g(z)\|_{\mathcal{L}_p[\overline{D}^{(\rho)}]}$$

the **mixed best approximation** of the function $f(z) \in \mathcal{L}_p[\overline{D}^{(\rho)}]$ on the set $\overline{D}^{(\rho)}$, and the expression

$$\omega(f; \delta; \overline{D}^{(\rho)})_p = \sup_{|\varphi| \leq \frac{\pi}{2}\left(1 - \frac{1}{\rho}\right)} \left\{ \sup_{|h| \leq \delta} \left( \int_{-\infty}^{\infty} \left| f[(t+h)e^{i\varphi}] - f(te^{i\varphi}) \right|^p |dz| \right)^{1/p} \right\}$$

* In the case $\rho = 1$, the set $\overline{D}^{(1)}$ is the whole real axis.

** In the case $\rho = 1$, the sets $\mathcal{L}_p[\overline{D}^{(1)}]$ and $\mathcal{L}_p^{(\sigma)}[\overline{D}^{(1)}]$ coincide respectively with the well-known classes of functions $L_p(-\infty, \infty)$ and $W_\sigma^{(p)}$ (see (2), p. 38).

with mixed modulus of continuity of the function $f(z) \in \mathcal{L}_p[\overline{D^{(\rho)}}]$.

Finally, let us denote by $B_p[G^{(\rho)}]$ the set of functions $f(z)$ $(z = re^{i\varphi})$, regular in the domain $G^{(\rho)}$ $(|\arg z| < \pi/2\rho,\ \rho \geqslant 1)$, having second partial derivatives with respect to $r$ and $\varphi$ in the closed domain $\overline{G^{(\rho)}}$ $(|\arg z| \leqslant \pi/2\rho,\ z = 0)$ and satisfying the conditions:

1. $|f(z)| \to 0$ as $|z| \to \infty$ $(z \in \overline{G^{(\rho)}})$.

2a.
$$\int_0^\infty |f(re^{i\pi/2\rho})|^p\, dr = M_1^p < +\infty.$$

2b.
$$\int_0^\infty |f(re^{-i\pi/2\rho})|^p\, dr = M_2^p < +\infty.$$

3. The integral $\int_0^\infty \frac{\partial^2}{\partial\varphi^2}|f(re^{i\varphi})|^p\, dr$ exists, and the equality
$$\frac{d^2}{d\varphi^2}\int_0^\infty |f(re^{i\varphi})|^p\, dr = \int_0^\infty \frac{\partial^2}{\partial\varphi^2}|f(re^{i\varphi})|^p\, dr$$
holds.

In the present work, applying the principle formulated above:

1) a relation is established in the form of inequalities between the quantities $A_\sigma(f; \overline{D^{(\rho)}})_p$ and $\omega(f; \sigma^{-1/\rho}; \overline{D^{(\rho)}})_p$ for functions $f(z) \in \mathcal{L}_p[\overline{D^{(\rho)}}]$;*

2) it is proved that $\omega(f; \delta; \overline{D^{(\rho)}})_p$ tends to zero as $\delta \to 0$.

3) an estimate of $A_\sigma(f; \overline{D^{(\rho)}})_p$ in the form of an inequality is found by means of $\sigma$ and $\rho$.

The solution of these questions is based on the following auxiliary lemmas:

**Lemma 1.** If $\lambda > 0$ is an even integer, then the entire function
$$H_\lambda(z) = \left\{\frac{1}{iz}\left[E_\rho(i\lambda^{-1/\rho}z) - E_\rho(-i\lambda^{-1/\rho}z)\right]\right\}^\lambda \quad (\rho \geqslant 1)$$
belongs to the set $\mathcal{L}_p^{(1)}[\overline{D^{(\rho)}}]$, where
$$E_\rho(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1 + n\rho^{-1})}$$
is the entire Mittag-Leffler function of order $\rho$ and type 1.

**Lemma 2.** For a function $f(z)$ $(z = re^{i\varphi})$ from the class $B_p[G^{(\rho)}]$, the formula
$$f(z) = \frac{e^{-i\pi/2\rho}}{2\pi i}\int_0^\infty \frac{f(ue^{-i\pi/2\rho})\, du}{ue^{-i\pi/2\rho} - z} - \frac{e^{i\pi/2\rho}}{2\pi i}\int_0^\infty \frac{f(ue^{i\pi/2\rho})\, du}{ue^{i\pi/2\rho} - z}, \qquad z \in G^{(\rho)}.$$

holds.

Moreover, we have

$$\lim_{r\to\infty} r^{k+1}\frac{\partial^k}{\partial r^k} f(re^{i\varphi}) = 0 \qquad (k=0,1,2,...).$$

**Lemma 3.** Let $K(\varphi) = a\sin\varphi + b\cos\varphi$, where $a, b$ are real numbers, and let the real function $g(\varphi)$ satisfy the conditions

$$r(\varphi) = g''(\varphi) + g(\varphi) \geqslant 0, \qquad g(0) = K(0), \qquad g(\alpha) = K(\alpha),$$

where $0 < \alpha < \pi$.

Then $g(\varphi) \leqslant K(\varphi)$ for $0 \leqslant \varphi \leqslant \alpha$.

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* A similar problem was considered in work (1) on the real axis in the metric of the space $C(-\infty,\infty)$, in work (3) in the strip $D = D(-\infty < x < \infty,\ -a < y < a)$ in the metric of the space $\mathcal{L}_p[D]$, and in work (4) on the set $\overline{D^{(\rho)}}$ in the metric of the space $C[\overline{D^{(\rho)}}]$.

We note that the proof of the convergence to zero of $\omega(f;\delta;\overline{D}^{(\rho)})_p$ as $\delta \to 0$ is based on the following assertion, which is an analogue of the classical Phragmén-Lindelöf theorem.

**Theorem 1.** For a function $f(z)$ of the class $B_p[G^{(\rho)}]$, the inequality

$$\int_0^\infty |f(re^{i\varphi})|^p\,dr \leq \frac{M_1^p + M_2^p}{2\cos\pi/2\rho}\cos\varphi + \frac{M_1^p - M_2^p}{2\sin\pi/2\rho}\sin\varphi \qquad \left(|\varphi| \leq \frac{\pi}{2\rho}\right),$$

holds, where $M_1^p, M_2^p$ are defined by equalities 2a and 2b.

We observe that, by virtue of this assertion, the norm of the function $f(z) \in \mathcal{L}_p[\overline{D}^{(\rho)}]$ in the domain $\overline{D}^{(\rho)}$ is estimated in terms of its same norm along the boundary straight lines $C_1 = C_1(\pi/2\rho)$ and $C_2 = C_2(-\pi/2\rho)$. Owing to this, the modulus of continuity of $f(z) \in \mathcal{L}_p[\overline{D}^{(\rho)}]$ over the domain $\overline{D}^{(\rho)}$ and its best approximation in the sense of the metric $\mathcal{L}_p[\overline{D}^{(\rho)}]$ by entire functions from the class $\mathcal{L}_p^{(\sigma)}[\overline{D}^{(\rho)}]$ are estimated* respectively in terms of the same moduli of continuity and the same best approximations along the boundary straight lines of the domain $\overline{D}^{(\rho)}$.

**Theorem 2.** If $f(z) \in \mathcal{L}_p[\overline{D}^{(\rho)}]$, then the quantities

$$\omega(f;\delta;C_1)_p = \sup_{\substack{|\xi|\le\delta\\ \xi\in C_1}} \left\{ \int_{C_1} |f(z+\xi)-f(z)|^p\,|dz| \right\}^{1/p},$$

$$\omega(f;\delta;C_2)_p = \sup_{\substack{|\xi|\le\delta\\ \xi\in C_2}} \left\{ \int_{C_2} |f(z+\xi)-f(z)|^p\,|dz| \right\}^{1/p}$$

and $\omega(f;\delta;\overline{D}^{(\rho)})_p$ are connected by the inequality

$$\omega(f;\delta;\overline{D}^{(\rho)})_p \le [\omega(f;\delta;C_1)_p + \omega(f;\delta;C_2)_p]\,\frac{1}{\sin\pi/2\rho},$$

where

$$C_1 = C\left[-\frac{\pi}{2}\left(1-\frac{1}{\rho}\right)\right], \qquad C_2 = C\left[\frac{\pi}{2}\left(1-\frac{1}{\rho}\right)\right].$$

Moreover,

$$\lim_{\delta\to 0}\omega(f;\delta;\overline{D}^{(\rho)})_p = 0.$$

**Theorem 3.** If $f(z)\in\mathcal{L}_p[\overline{D}^{(\rho)}]$, then there exists an entire function $g_0(z)$ from the class $\mathcal{L}_p^{(\sigma)}[\overline{D}^{(\rho)}]$ for which the inequalities

$$\omega(f;\sigma^{-1/\rho};D^{-(\rho)})_p \le 2A_\sigma(f;\overline{D}^{(\rho)})_p + \omega(g_0;\sigma^{-1/\rho};D^{(\rho)})_p,$$

$$A_\sigma(f;\overline{D}^{(\rho)})_p \le B\omega(f;\sigma^{-1/\rho};\overline{D}^{(\rho)})_p,$$

hold, where $B$ is a constant independent of $\sigma,\rho$.

**Corollary 1.** If $\omega(f;\sigma^{-1/\rho};\overline{D}^{(\rho)})_p \to 0$ as $\sigma\to\infty$, then $A_\sigma(f;D^{(\rho)})_p \to 0$ as $\sigma\to\infty$, and conversely.

**Corollary 2.** For every function $f(z)\in\mathcal{L}_p[\overline{D}^{(\rho)}]$ there exists an entire function from $\mathcal{L}_p^{(\sigma)}[\overline{D}^{(\rho)}]$ such that

$$\lim_{\sigma\to\infty} A_\sigma(f;\overline{D}^{(\rho)})_p = 0.$$

**Theorem 4.** If $f(z)$, together with all derivatives up to order $m$, belongs to $\mathcal{L}_p[\overline{D}^{(\rho)}]$, then

$$A_\sigma(f;\overline{D}^{(\rho)})_p \leq \frac{C_m}{\sigma^{m/\rho}},$$

where $C_m$ does not depend on $\sigma$.

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* The authors have considered similar problems in a horizontal and in a vertical strip of the given width, bounded by straight lines.

*Note: Figure translations are in progress. See original paper for figures.*