ESTIMATES OF SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS OF HIGHER ORDERS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.946.9

MATHEMATICS

L. A. GUSAROV

ESTIMATES OF SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS OF HIGHER ORDERS

(Presented by Academician I. G. Petrovskii, May 19, 1965)

In this note we derive Schauder-type estimates for solutions of the first boundary value problem for parabolic equations of order $2p$, $p > 1$. For a second-order parabolic equation such estimates were obtained by A. Friedman (^{1}). Friedman' s proof is based on the use of an explicit expression for the fundamental solution of the heat equation, on the possibility of transforming a second-order parabolic equation with constant coefficients to the heat equation, and on the possibility of representing the Green' s function as the difference of two fundamental solutions, i.e., on properties that do not hold for equations of order higher than two. It turned out that one can do without these properties if one improves the estimate of the derivative of order $(2p-1)$ of the fundamental solution and uses the Green' s function constructed by V. P. Mikhailov (^{2}). To avoid cumbersome exposition, we consider the plane case and then show that the proof also goes through in the case of $n > 1$ spatial variables.

Consider, in the closure $\overline{D}$ of a bounded domain D in the (x, t)-plane, the equation, parabolic in the sense of Petrovskii,

$$L(x, t, D_x, D_t)u \equiv [A(x, t, D_x) - D_t]u = f(x, t), \tag{1}$$

where

$$A(x, t, D_x) \equiv A_{2p}(x, t)D_x^{2p} + \sum_{k<2p} A_k(x, t)D_x^k, \quad k \geq 0, \quad p > 1$$

are integers, $D_x = \partial/\partial x$, $D_t = \partial/\partial t$.

We define the distance between points $P(x, t)$ and $Q(x', t')$ in D by the formula

$$d(P, Q) = \left(|x - x'|^2 + (t - t')^{2/2p}\right)^{1/2}. \tag{2}$$

Suppose that the boundary of the domain D consists of a part Γ_0 of the line $t=0$, a part Γ_1 of the line $t=t_1$, and a lateral curve $\overline{B}$ satisfying the following property:

For each point $P \in \overline{B}$ there exists a circle V with center at P such that the part of $\overline{B}$ lying in V can be represented in the form

$$x = h(t), \tag{3}$$

where h has a Hölder-continuous derivative with respect to t (with exponent $0 < \beta < 1$).

Introduce the notation

$$\Gamma_0 \cup \overline{B} = \Gamma; \tag{4}$$

$$[z]_j^D = \sup_{P \in D} |D_x^j z(P)|, \qquad [z]_{j+\beta}^D = H[D_x^j z],$$

$$|z|_\beta \equiv |z|_\beta^D = \sup_D |z| + H[z], \qquad |z|_l^D = \sum_{j=0}^{l} [z]_j^D, \tag{5}$$

$$|z|_{l+\beta} \equiv |z|_{l+\beta}^D = \sup_D \left(\sum_{j=0}^{l} |D_x^j z| + |D_t z| \right) + \sum_{j=0}^{l} H[D_x^j z] + H[D_t z],$$

$H(z)$ is the Hölder constant for $z(P)$ in D, i.e., the least of the constants C satisfying the condition

$$\frac{|z(P) - z(Q)|}{d(P,Q)^\beta} \le C$$

for any P and Q from D; l and j are nonnegative integers.

Theorem. Let $u(x,t)$ be a solution of equation (1) in D, taking the boundary values

$$u(x,t) = \varphi(x) \quad \text{on } \Gamma_0; \tag{6}$$

$$\partial^j u(x,t)/\partial n^j = \psi_j(x,t) \quad \text{on } \overline{B}, \tag{7}$$

$j = 0, 1, \dots, p-1$; $\partial/\partial n$ is differentiation in the normal direction; φ and ψ_j satisfy compatibility and smoothness conditions such that there exists a function

$\Phi(x,t)$, defined on $\overline{D}$, coinciding with φ on Γ_0, whose normal derivatives to $\overline{B}$ coincide with ψ_j, and the condition

$$|\Phi|^* = \inf |\Phi|^D_{2p+\beta} < +\infty, \tag{8}$$

is fulfilled, where the infimum is taken over all possible extensions $\Phi(x,t)$.

Assume that the coefficients of equation (1) and $f(x,t)$ satisfy in $\overline{D}$ the condition

$$|A_{2p}|_\beta \leq K_1, \qquad |A_k|_\beta < K_1 \quad \text{for } k < 2p, \qquad |f|_\beta < K_1 \tag{9}$$

and that $u(x,t)$ has Hölder-continuous (with exponent $0 < \beta < 1$) derivatives with respect to x up to order $2p$ inclusive in D.

Then

$$|u|_{2p+\beta} \leq K \left(\sup_D |u| + |\Phi|^* + |f|_\beta \right), \tag{10}$$

where K depends only on K_1, the parabolicity constant, and the dimensions of D, but does not depend on $u(x,t)$.

The proof of the theorem is carried out with the aid of three lemmas.

Lemma 1. Let $G(x,\xi;t,\tau)$, $t > \tau$, be the fundamental solution of the equation

$$(-1)^{p-1} D_x^{2p} u - D_{tu} = 0, \tag{11}$$

where $p > 1$ is an integer. Then the estimate

$$|D_x^{2p-1} G| \leq C|x-\xi|/|t-\tau|^{(2p+1)/2p} \times$$

$$\times \exp\{-c_1 |x-\xi|^{2p/(2p-1)}/|t-\tau|^{1/(2p-1)}\} \tag{12}$$

is valid.

Proof

$$G(x,\xi;t,\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-\xi)\alpha} e^{-\alpha^{2p}(t-\tau)} \, d\alpha \quad \text{for } t > \tau,$$

$$G(x,\xi;t,\tau) = 0 \quad \text{for } t \leq \tau.$$

For $t > \tau$, by integration by parts we obtain

$$D_x^{2p-1}G = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-\xi)\alpha}(i)^{2p-1}\alpha^{2p-1}e^{-\alpha^{2p}(t-\tau)}\,d\alpha =$$

$$= \frac{(-1)^p(x-\xi)}{2\pi\cdot 2p(t-\tau)}\int_{-\infty}^{+\infty} e^{i(x-\xi)\alpha}e^{-\alpha^{2p}(t-\tau)}\,d\alpha = C\,\frac{(x-\xi)}{(t-\tau)}\,G. \qquad (13)$$

From (13) and the estimates of the fundamental solution (3) we obtain (12). Estimate (12) is also valid for the fundamental solution $\widetilde{G}$ of the equation adjoint to (11), if $x=\xi,\ t=\tau$ is a singular point of $\widetilde{G}$.

Introduce the notation: R_1 is the interval $-1 < x < 1$; S_τ is the rectangle $-1 < x < 1,\ 0 < t < \tau$; $\overline{S}_\tau$ is the closure of S_τ.

Lemma 2. Let $f(x,t)$ be defined and Hölder continuous (with exponent $0 < \beta < 1$) in $\bar{S} = \bar{S}_1$, and let $f(\pm 1, 0) = 0$. Then $u_t(\xi,\tau)$, where

$$u(\xi,\tau) = \int_0^\tau \int_{R_1} f(x,t)\,\widetilde{G}(x,\xi;t,\tau)\,dx\,dt \qquad (14)$$

will be Hölder continuous in S, and the inequality

$$|u_t(P) - u_t(Q)|/d(P,Q)^\beta \le AH[f], \qquad (15)$$

holds, where A depends only on β.

The **proof** is based on Lemma 1 and is carried out analogously to the proof of the fundamental lemma in (1).

Lemma 3. Let $P(\xi_0,\tau_0)$ be a fixed point in the plane (x,t); let R be the interval $\xi_0 - \tilde{d} < x < \xi_0 + d$, where $0 \le \tilde{d} < d$, $d > 0$, $\tau \le \tau_0 \le d^{2p}$, $S_{\tau,d} = R\times(0 < t < \tau)$, $S_{\tau_0 d} = S_d$.

If $f(x,t)$ satisfies a Hölder condition (with exponent β) in the closure $\bar{S}_d$, $u(x,t)$ is defined in $\bar{S}_d$, has there continuous derivatives $D_x^{2p}u$, D_{tu}, satisfies the equation

$$L(u) = (-1)^{p-1}D_x^{2p}u - D_{tu} = f(x,t), \qquad (16)$$

is equal to zero together with its derivatives up to order $(p-1)$ with respect to x on the line $x = \xi_0 - \tilde{d}$, and is equal to zero on the line $t = 0$, then D_x^j, $0 \le j \le 2p$, satisfies a Hölder condition (with exponent β) in $\bar{S}_d$, and there exists a constant K, depending only on β, such that for $j = 0, 1, \dots, 2p$

$$|D_x^j u(P)| \le d^{-j}K\sup_{S_d}|u| + d^{2p-j}K\sup_{S_d}|f| + d^{2p-j+\beta}KH_{P,S_d}[f] \equiv KI; \qquad (17)$$

$$d^\beta |D_x^j u(P) - D_x^j(Q)|/d(P,Q)^\beta = KI + Kd^{2p-j+\beta} H[f], \tag{18}$$

where

$$Q \in S_d, \qquad d(P,Q) < \tfrac{1}{4} d, \qquad H_{P,S_d}[f] \equiv \sup_{Q \in S_d} \frac{|f(P) - f(Q)|}{d(P,Q)^\beta},$$

$$H[f] \equiv \sup_{\widetilde{P} \in S_d} H_{\widetilde{P},S_d}[f].$$

The **proof** is carried out with the help of Lemmas 1 and 2 and the representation of the solution $u(\xi,\tau)$ at points $(\xi,\tau) \in S_{d/2}$ in the form

$$u(\xi,\tau) = -\int_0^\tau \int_R f(x,t)\varphi(x,t)\widetilde{G}(x,\xi;t,\tau)\, dx\, dt +$$

$$+ \int_0^\tau \int_R u(x,t) \widetilde{L}\left[\varphi(x,t)\widetilde{G}(x,\xi;t,\tau)\right]\, dx\, dt,$$

where $\widetilde{G}(x,\xi,t,\tau)$ is the Green' s function for the equation $\widetilde{L}(u) = 0$, adjoint to the equation $L(u) = 0$:

$$\widetilde{G}(x,\xi;t,\tau) = \begin{cases} \widetilde{G}(x,\xi;t,\tau) + G_0(x,\xi;t,\tau), & \text{for } t < \tau, \\ 0, & \text{for } t \geq \tau; \end{cases}$$

$\widetilde{G}$ is the fundamental solution of the equation $\widetilde{L}(u) = 0$ with pole $\xi = x$, $t = \tau$; G_0 is a solution regular in S_d of the equation $\widetilde{L}(u) = 0$, satisfying the conditions

$$D_x^j G_0\big|_{x=\xi_0 - \tilde{d}} = -D_x^j \widetilde{G}\big|_{x=\xi_0 - d}, \qquad D_x^j G_0\big|_{x=\xi_0 + d} = -D_x^j \widetilde{G}\big|_{x=\xi_0 + d}.$$

The Green' s function is constructed in (2):

$$\varphi(x,t) = \begin{cases} 1, & \text{if } (x,t) \in S_{d/2}\ (\xi_0 - \tilde{d} \leq x \leq \xi_0 + d/2), \\ 0, & \text{if } (x,t) \in S_d - S_{3d/4}, \end{cases}$$

and is infinitely differentiable in S_d.

The proof of the theorem is then carried out entirely analogously to the proofs in (1, 4), i.e., by the classical method developed by Schauder.

Remark 1. In the case of $n > 1$ spatial variables, for the fundamental solution G of an equation with constant coefficients

$$\frac{\partial u}{\partial t} = \sum_{k_1+k_2+\dots+k_n=2p} A_{k_1\dots k_n} D^{k_1\dots k_n}_{x_1\dots x_n} u$$

one obtains the relation

$$D^{(2p-1)}_{x_j} G = K_2 \frac{x_j - \xi_j}{t-\tau} G + K_3 \sum B_k D^{(2p-1)}_{x_{k_1}\dots x_{k_n}} G,$$

where K_2 and K_3 are constants, and the last sum does not contain the derivative $D^{(2p-1)}_{x_j}$, which makes it possible to prove analogues of Lemmas 1, 2, 3 and of the theorem. In this case, one must require of the lateral surface B that the part of it falling inside the sphere V be representable, for some j, in the form

$$x_j = h(t, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \tag{3'}$$

where h has Hölder-continuous (with exponent β) derivatives with respect to the spatial variables up to order $2p$ inclusive, and with respect to t—of first order.

Remark 2. In (5), estimates in Hölder norms for solutions of systems including parabolic systems in the sense of Petrovskii were published without proof, but only for the cylindrical domain $D = \Omega \times [0, T]$. In our theorem the domain D is of a more general form.

Received
17 V 1965

CITED LITERATURE

1. A. Friedman, J. Math. and Mech., 7, No. 5, 771 (1958).
2. V. P. Mikhailov, DAN, 132, No. 2, 291 (1960).
3. S. D. Eidelman, Matem. sborn., 33 (75), 359 (1953).
4. A. Friedman, J. Math. and Mech., 7, No. 3, 393 (1958).
5. V. A. Solonnikov, DAN, 157, No. 1 (1964).

Note: Figure translations are in progress. See original paper for figures.