ON THE THEORY OF LOCALLY FINITE GROUPS

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Abstract

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§ 1. In accordance with the work ([1]), a subgroup H of a group G is called **2-infinitely isolated** in G if, from the fact that the centralizer of a nonidentity element of H in G contains at least one involution from G and has infinite intersection with H, it follows that it belongs to the subgroup H.

In the present note we study a locally finite group possessing a maximal complete abelian 2-subgroup whose normalizer is a 2-infinitely isolated subgroup. The theorem proved makes it possible to reduce the solution of the minimality problem ([2,3]) for locally finite groups to the case when its Sylow 2-subgroup is finite.

§ 2. **Definition.** A subgroup H of a group G is **2-mutually simple with all its conjugates** if it has intersection with any subgroup conjugate to it in G, distinct from it, in a subgroup containing no 2-elements. Analogously one may introduce the notion of π-mutual simplicity of a subgroup in a group, where π is a certain set of prime numbers. As in ([1]), an involution with finite centralizer in the group will be called **almost regular**.

Theorem. *If in a locally finite group G the normalizer N of some maximal complete abelian 2-subgroup A is a 2-infinitely isolated subgroup, then one of the following assertions holds:*

1) *some almost regular involution in H is finitely approximable in the group G;*
2) *the group G is isomorphic to a group of type $PGL(2, K)$ over a field K of odd characteristic;*
3) *$N = G$;*
4) *the group G is a Frobenius group.*

If the subgroup N possesses an almost regular involution, then, in view of the main result of ([1]), one of assertions 1) and 2) holds. Further, the proof of the theorem is based on the assumption that, in a group satisfying the conditions of the theorem, none of the possibilities 1)–4) is realized. This part of the proof rests on the following lemmas.

Lemma 1. *All involutions in the group G are conjugate. All involutions from N are conjugate in N and are contained in the subgroup A.*

Lemma 2. *Let G be a certain periodic group, H its proper subgroup, coinciding with its normalizer in G, and 2-mutually simple with all its conjugates.*

If in the subgroup H there exists an elementary abelian 2-subgroup of rank 2, then for some element $h \neq 1$ from H of odd order there is an involution $t \in G$ such that $t \notin H$ and $t^{-1}ht = h^{-1}$.

Proof. From the condition of the lemma follows the existence of an involution $t_1 \in G$ not belonging to the subgroup H. Further, by the condition of the lemma, the subgroup H has two involutions i and j, commuting with each other, and $i \neq j$.

Consider two elements $a = it_1$, $b = jt_1$. They are, evidently, of odd order, since otherwise, by the condition of the lemma, $t_1 \in H$, which is impossible. It follows that the involution t_1 is conjugate to i in the subgroup $\{a\}\lambda\{t_1\}$ and to the involution j in the subgroup $\{b\}\lambda\{t_1\}$. Consequently, for some natural numbers k and r we shall have $a^{-k}ia^k = t_1$, $b^{-r}jb^r = t_1$. Hence $a^{-k}ia^k = b^{-r}jb^r$, $b^r a^{-k} i a^k b^{-r} = j$.

By the condition of the lemma, $h_1 = a^k b^{-r} \in H$. Since $i \neq j$, $h_1 \neq 1$. From $h_1 = a^k b^{-r}$ we obtain $a^k = h_1 b^r$. We transform the left and right sides of this equality by means of the involution t_1: $t_1^{-1}a^k t_1 = t_1^{-1}h_1 t_1 t_1^{-1} b^r t_1$, or $a^{-k} = t_1^{-1}h_1 t_1 b^{-r}$. On the other hand, $a^{-k} = b^{-r}h_1^{-1}$. From the last two equalities we obtain $b^{-r}h_1^{-1} = t_1^{-1}h_1 t_1 b^{-r}$, or $b^r t_1^{-1} h_1 t_1 b^{-r} = h_1^{-1}$. The element $i_1 = brt_1^{-1}$ is an involution, since $i_1 \in \{b^r\}\lambda\{t_1\}$.

Let $i_1 \notin H$. Then the order of the element h_1 is odd. Indeed, suppose that in the subgroup $\{h_1\}$ there exists an involution i_2. Evidently, $i_1 i_2 = i_2 i_1$. Since $i_2 \in H$, by the conditions of the lemma, $i_1 \in H$, which is impossible. It follows that the pair of elements h_1, i_1 satisfies all the requirements of the lemma.

Let $i_1 \in H$. Since $j \in H$ and $j = bt_1^{-1}$, we have $i_1 j^{-1} = brt_1^{-1} t_1 b^{-1} = a^{k-1} \in H$. If $i_2 = i$, $a^{k-1} \neq 1$, and the pair of elements a^{-1}, t_1 satisfies all the requirements of the lemma.

Let $i_1 = j$, i.e., $r = 1$. Then consider the involution $i_2 = a^k t_1^{-1} \in \langle h_1 \rangle \lambda \langle i_1 \rangle$. Since $i_2 \in H$, $i \in H$ and $i = at_1^{-1}$, it follows that $i_2 i^{-1} = a^k t_1^{-1} t_1 a^{-1} = a^{k-1} \in H$. If $i_2 \neq i$, $a^{k-1} \neq 1$, and the pair of elements a^{k-1}, t_1 satisfies all the requirements of the lemma.

It remains to consider the case when $k = r = 1$. Since $h_1 = a^k b^{-r}$, for $k = r = 1$ we have $h_1 = ab^{-1} = itt^{-1}j^{-1} = ij^{-1}$. By the condition of the lemma, the involutions i and j commute and $i \neq j$, but then the equality $h_1^{-1} i h_1 = j$ is impossible. The contradiction obtained indicates that at least one of the numbers k and r is not congruent to unity modulo the corresponding order of one of the elements a and b, and, as was shown above, in this case the assertion of the lemma is true. The lemma is proved.

Lemma 3. *The subgroup N is a semidirect product of a Hall subgroup T and some locally cyclic subgroup $D \neq 1$, and every nontrivial element of D induces a regular automorphism in T.*

The proof of the lemma is based on the definition of a 2-infinitely isolated subgroup, Lemmas 1 and 2, and Burnside' s theorem ([4]).

Lemma 4. *If the group G satisfies the conditions of Lemma 3, then a Sylow 2-subgroup S of the group G is elementary abelian, and the centralizer of any involution from S in G is contained in the centralizer of the subgroup S in G.*

In the proof of the lemma, Lemmas 1 and 3 are used, together with Thompson' s theorem on a finite group possessing a regular automorphism of prime order ([5]), and the Higman–Hall theorem on the length of the upper central series of a finite nilpotent group possessing an automorphism of prime order ([6]).

The concluding part of the proof of the theorem is based on a result of W. Feit on the abstract characterization of groups of type $SL(2, 2^a)$, $a > 1$ ([7]).

We turn to the discussion of the minimality problem in locally finite groups. The problem is formulated as follows ([3]): is a locally finite group with the minimality condition for subgroups extremal?

Let G be a locally finite group with the minimality condition for subgroups, not extremal. As is not difficult to show, without violating the generality of the argument one may assume that G is an infinite simple group and all proper subgroups of G are extremal. Since a finite group of odd order is solvable ([8]), and a locally solvable group with the minimality condition for subgroups is extremal (theo-

Chernikov' s theorem ([9]), we may restrict ourselves to considering the case when the group G has involutions.

We shall show that a Sylow 2-subgroup S of G cannot be infinite. Let S' be a maximal complete abelian subgroup of S. As is easy to see, the subgroup S' and its normalizer $N_G(S')$ in G satisfy all the conditions of the theorem formulated above, and therefore, for the group G, one of assertions 1)–4) of this theorem holds. Since G is an infinite simple group, it is isomorphic to a group of type $PGL(2, K)$ over a field K of odd characteristic. However, an infinite group of type $PGL(2, K)$ does not satisfy the minimality condition for subgroups. The contradiction obtained shows that the subgroup S cannot be infinite.

Thus we have proved:

The minimality problem for subgroups in a locally finite group is sufficiently solved for the case when its Sylow 2-subgroup S is finite and $S \neq 1$.

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Note: Figure translations are in progress. See original paper for figures.