INVESTIGATION OF THE CAUCHY PROBLEM BY THE METHOD OF WEAK APPROXIMATION

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.919

MATHEMATICS

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INVESTIGATION OF THE CAUCHY PROBLEM BY THE METHOD OF WEAK APPROXIMATION

(Presented by Academician L. V. Kantorovich on 16 VII 1965)

In the paper ([1]) the question of convergence of the method of fractional steps in differential form was investigated in solving a well-posed Cauchy problem in a Banach space. In the present paper this question is considered without the assumption of well-posedness of the original Cauchy problem, and it is shown that its well-posedness is a consequence of the uniform well-posedness of a certain auxiliary Cauchy problem. The proposed method of investigation is based on the idea of weak approximation of differential operators.

Definition 1. A family of functions $F_\tau(x,t)$ **weakly approximates with respect to** t the function $F(x,t)$ for $0 < t < T$ and $x \in \Omega$, $\Omega \subset E^m$, if

$$\int_{t_1}^{t_2} [F_\tau(x,s) - F(x,s)]\, ds = \delta(x, t_1, t_2, \tau) \tag{1}$$

and $\|\delta\| \to 0$ as $\tau \to 0$ for any fixed admissible t_1, t_2. A family of linear differential operators $L_\tau(x,t)$ weakly approximates with respect to t the operator $L(x,t)$, if weak approximation holds for the coefficients.

The integral in (1) should be understood in the Riemann sense, except for the case of the space $L_q(\Omega)$, where the Lebesgue integral is also used.

We investigate the well-posedness of the formulation of the Cauchy problem

$$\partial u/\partial t = Lu, \quad 0 < t < T; \qquad u|_{t=0} = u_0, \quad u_0 \in B \tag{I}$$

in the Banach space B. With respect to the space B we assume: a) B is a space of functions of m real variables $x_1, x_2, \dots, x_m$, which may depend parametrically on t; b) in B the operations of differentiation with respect to the spatial variables $\partial/\partial x_k$ and with respect to time $\partial/\partial t$ are defined and are closed with respect to strong convergence; c) sufficiently smooth functions are dense in B.

Let the operator L be representable as a sum

$$L = L_1 + L_2 + \dots + L_p$$

of linear operators of the form

$$L_i = \sum_{k_1,\dots,k_m} a^i_{k_1\dots k_m}(x_1,\dots,x_m,t)\,\frac{\partial^{k_1+\dots+k_m}}{\partial x_1^{k_1}\cdots\partial x_m^{k_m}}, \tag{2}$$

where $a^i_{k_1\dots k_m}$ are real functions, bounded and continuous in t in the uniform topology. If the function $a(x_1,\dots,x_m,t)$ has all the derivatives entering into the expressions (2), then we shall say that it has derivatives up to order L; if such a differentiation procedure can be repeated j times, then we shall say that $a(x_1,\dots,x_m,t)$ has derivatives up to order $(L)^j$. Let $a^{k_1}_{\dots k_m}$ have derivatives with respect to the spatial variables up to order $(L)^j$. Then, formally differen-

differentiating the equation and the initial data of problem (I), we obtain the problem

$$\partial u^j/\partial t = L^j u^j, \quad 0 < t < T; \qquad u^j\big|_{t=0} = u^j_0, \tag{I^j}$$

where $L^j = L^j_1 + L^j_2 + \dots + L^j_p$.

Here u^j denotes the vector function consisting of u and of the derivatives of u with respect to the spatial variables up to order $(L)^j$; L^j_i is the operator naturally corresponding to the operator L_i, if one adheres to the rule

$$\frac{\partial M v}{\partial x_k} = \frac{\partial M}{\partial x_k} v + M\frac{\partial v}{\partial x_k},$$

where v is a component of the vector u^j; M is a linear differential operator, and $\partial v/\partial x_k$ is understood as a component of the vector u^j. As the norm of u^j we choose the Euclidean vector norm, denoting by B_j the set of functions in B for which this norm is finite.

Let there be associated with the parameter θ a family of problems

$$\partial u/\partial t = Lu, \quad 0 \le \theta < t < T; \qquad u\big|_{t=0} = u_0, \quad u_0 \in B. \tag{3}$$

Definition 2. Problem (I) is **uniformly well-posed in** B **for** $0 < t < T$ if: a) problem (3) is uniquely solvable for a set of functions u_0 dense in B, i.e. $u(t) = S(t,\theta)u_0$; b) the transition operator $S(t,\theta)$ has the following properties:

$$S(t_2,\theta) = S(t_2,t_1)S(t_1,\theta), \qquad 0 \le \theta < t_1 < t_2 < T; \tag{4}$$

$$\|S(t_2,t_1)\| \le e^{\alpha(t_2-t_1)}, \qquad 0 \le t_1 < t_2 < T; \tag{5}$$

$$\|S(t,\theta)u_0 - u_0\| \underset{t\to\theta}{\longrightarrow} 0, \qquad 0 \le \theta < t < T. \tag{6}$$

If the operator $S(t,\theta)$ satisfies conditions (4)–(6) in B, we shall also say that it satisfies the conditions of uniform well-posedness. In terms of semigroup theory, an operator $S(t,\theta)$ satisfying conditions (4)–(6) belongs to the class (C_0) ([2]).

To problem (I) we associate the auxiliary (factorized) problem

$$\partial u_\tau/\partial t = L_\tau u_\tau, \quad 0 < t < T; \qquad u_\tau\big|_{t=0} = u_0, \quad u_0 \in B, \tag{I_τ}$$

where

$$L_\tau = \sum_{i=1}^{p} \alpha_i(t,\tau) L_i,$$

$$\alpha_i(t,\tau) = \begin{cases} p, & t \in \left(\left(n + \dfrac{i-1}{p}\right)\tau, \left(n + \dfrac{i}{p}\right)\tau\right], \\ 0 & \text{in the remaining cases.} \end{cases}$$

The properties of the solutions of the family of problems (I_τ) are completely determined by the properties of the solutions of the problems

$$\partial u^j/\partial t = pL_i^j u^j, \qquad 0 < t < T; \tag{I_i^j}$$

$$u^j\big|_{t=0} = u_0^j, \qquad u_0 \in B_j, \qquad j = 0, 1, \dots, \qquad u^0 \equiv u, \qquad B_0 \equiv B.$$

It is not difficult to see that the transition operator of problem (I_τ) is the product of the transition operators of the problems (I_i^0),

$$S_\tau(t+\tau,t) = S_p\left(t+\tau, t + \frac{p-1}{p}\tau\right) \cdots S_1\left(t + \frac{\tau}{p}, t\right) \quad \text{for } t = n\tau.$$

Theorem 1. *If the problems* (I_i^0), (I_i^1), (I_i^2) *are uniformly well-posed, then* $u_\tau(t)$ *converges uniformly in* t *to the function* $u(t) = S(t,0)u_0$ *as* $\tau \to 0$, *and the transition operator* $S(t,0)$ *satisfies the conditions of uniform well-posedness.*

Theorem 2. *If the problems* (I_i^0), (I_i^1), (I_i^2), (I_i^3) *are uniformly well posed, then* $u_\tau(t)$, *together with its derivatives with respect to* x_k *up to order* L, *converges* *strongly, uniformly in t, to the function $u(t)$ as $\tau \to 0$. The limiting function $u(t)$ has a derivative with respect to t and is a solution of problem (I).*

In the terms of semigroup theory, Theorems 1 and 2 give sufficient conditions for the closure of a certain restriction of the sum of the infinitesimal generators of semigroups of class (C_0) also to be an infinitesimal generator of a semigroup of class (C_0). The first results on the identification of the class of operators that are infinitesimal generators of semigroups are due to Hille ([3]) and Yosida ([4]). The further development of this theory and its application to the study of the abstract Cauchy problem are set forth in the monograph ([2]). Semigroups with a t-dependent infinitesimal generator and the Cauchy problem associated with them were studied by P. Lax and R. Richtmyer ([5]). A survey of research on the abstract Cauchy problem is given in ([6]).

Let B^1_θ be the space with norm

$$\|f\|_{B^1_\theta} = \int_\theta^T \|f\|_B \, dt.$$

To the inhomogeneous problem

$$\partial u/\partial t = Lu + f, \quad 0 < t < T; \qquad u|_{t=0} = u_0, \quad u_0 \in B; \qquad f \in B^1(I)$$

and to the parameter θ we associate the family of inhomogeneous problems

$$\partial u/\partial t = Lu + f, \quad 0 \le \theta < t < T; \qquad u|_{t=\theta} = u_0, \quad u_0 \in B; \qquad f \in B^1_\theta. \quad (7)$$

If, for given u_0 and f, problem (7) is solvable, we introduce the notation

$$v(t) = u(t) - S(t, \theta) u_0,$$

where $S(t, \theta)$ is the evolution operator of problem (3).

Definition 3. Problem (I) is **well posed with respect to the right-hand side** for $0 < t < T$ if $v(t)$ depends continuously on f.

Let $u(t)$ be a solution of problem (I). The following holds.

Theorem 3. *If: a) $u(0) = u_0 \in B_1$; b) $L_i u$ are uniformly continuous in t; c) the problems (I_i^0), (I_i^1) are uniformly well posed; d) the problems (I^0) are well posed with respect to the right-hand side, then $u(t)$ is the unique solution of problem (I) satisfying a) and b), and $u_\tau(t)$ converges to $u(t)$ uniformly in t.*

In the case when $B \equiv L_q(\Omega)$, $q > 1$, $\Omega \subset E^m$, we shall assume that $u^1(t)$ includes at least all first derivatives of $u(t)$ with respect to the spatial variables. We shall say that $u(t)$ is a **smooth function** if it has all derivatives entering into the equation of problem (I), belonging to $L_q(\Omega^1)$, $\Omega^1 = \Omega \times (0, T)$.

Theorem 4. *If: a) the problems (I_i^0), (I_i^1) are uniformly well posed; b) the problems (I^0) are well posed with respect to the right-hand side, then $u_\tau(t)$*

converges weakly as $\tau \to 0$, uniformly in t, and every smooth limiting function $u(t)$ is a solution of problem (I); in the case $u_0^1 \in L_q(\Omega)$, $u(t)$ is the smooth and unique solution of problem (I).

Finally, if $m = p$, $q = 2$, $L_i = A_i(x,t)\partial/\partial x_i$, where A_i are symmetric matrices, continuous in Ω^1 together with their first derivatives with respect to the spatial variables, and Ω^1 is the usual cone of dependence, then the following holds.

Theorem 5. *The problems (I), (I_i^0) are uniformly well posed. The function $u_\tau(t)$ converges uniformly in t, as $\tau \to 0$, to the solution of problem (I).*

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Received
10 VII 1965

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Note: Figure translations are in progress. See original paper for figures.