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Abstract

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ON THE NORMAL SOLVABILITY OF THE DIRICHLET PROBLEM FOR A SYSTEM OF ELLIPTIC TYPE OF A. V. BITSADZE

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Among elliptic boundary-value problems for which the Noether property is violated, the most typical is the Dirichlet problem for the system of A. V. Bitsadze

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} - 2\frac{\partial^2 u_2}{\partial x\,\partial y} = F_1(x,y),$$

$$\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} + 2\frac{\partial^2 u_1}{\partial x\,\partial y} = F_2(x,y); \tag{1}$$

$$u_1\big|_\Gamma = f_1(s), \qquad u_2\big|_\Gamma = f_2(s). \tag{2}$$

In the work ([1]) it was shown that the homogeneous problem (1^0)—(2^0) ($F_1 \equiv F_2 \equiv 0,\ f_1 \equiv f_2 \equiv 0$) has in a disk an infinite number of linearly independent solutions, while the nonhomogeneous problem for the same domain is normally solvable in the sense of Hausdorff ([3]).

The aim of the present work is to study the nature of the violation of the Noether property of problem (1)–(2) for a broad class of noncircular domains.

In complex form, problem (1)–(2) can be written as

$$\partial^2 w/\partial \bar z^2 = F(z), \tag{1'}$$

$$w\big|_\Gamma = f(s), \tag{2'}$$

where $w = u_1 + iu_2,\ F = F_1 + iF_2,\ f = f_1 + if_2$.

It is assumed that the derivative $\partial^2/\partial\bar z^2$ is understood in the sense of S. L. Sobolev, $F(z) \in L_p(\mathcal{D})$, $p > 1$, $\mathcal{D}$ is a simply connected domain with Lyapunov boundary Γ, and the boundary condition $(2')$ is satisfied almost everywhere, with $f(s) \in L_2(\Gamma)$.

Without loss of generality, one may assume that $F(z) \equiv 0$, i.e.

$$\partial^2 u/\partial \bar{z}^2 = 0; \tag{I}$$

$$u\big|_\Gamma = f(s). \tag{II}$$

This is easily verified by replacing the sought solution by

$$w = u + \frac{1}{\pi^2} \iint_D \frac{d\xi_1\, d\eta_1}{t_1 - z} \iint_D \frac{F(\xi, \eta)}{t - t_1}\, d\xi\, d\eta.$$

The general solution of equation (I), as is known ([2]), has the form

$$u(z) = \bar{z}\varphi(z) + \psi(z), \tag{3}$$

where $\varphi(z)$, $\psi(z)$ are arbitrary holomorphic functions in $\mathcal{D}$.

Definition. A solution $u(z)$ of equation (I) will be called **regular** if the functions $\varphi(z)$ and $\psi(z)$ in representation (3) belong to the class $E_2(\mathcal{D})$ ([4]).

We shall call problem D the problem of finding regular solutions in $\mathcal{D}$ of equation (I), satisfying almost everywhere the boundary condition (II).

Below we study both the homogeneous problem D_0 ($f \equiv 0$) and the nonhomogeneous problem D.

Problem (I)–(II) is easily reduced to a Fredholm integral equation of the first kind. Therefore the violation of Noetherianity for this problem and for a Fredholm integral equation of the first kind has the same character.

Let $z(\omega)$ be a holomorphic function in the unit disk S, continuous in the closed domain $S + \gamma$; let $\{\varphi_i(\omega)\}$ be the system of functions determined from the expansion of $z(\omega)$ into a power series near zero

$$z(\omega) = \sum_{k=0}^{i-1} a_k \omega^k + \omega^i \varphi_i(\omega). \tag{4}$$

Lemma 1. In the system $\{\varphi_i(\omega)\}$, either all the functions are linearly independent, or among them there exists only a finite number m of linearly independent functions $\varphi_1(\omega), \varphi_2(\omega), \dots, \varphi_m(\omega)$. The latter case occurs if and only if $z(\omega)$ is a rational function.

Let now $z(\omega)$ be a conformal mapping of the unit disk $|\omega| \leq 1$ onto the domain $\mathcal{D}$.

Theorem 1. Among simply connected domains bounded by analytic curves, the homogeneous problem D_0 has nonzero regular solutions if and only if $z(\omega)$ is a rational function.

This fact, under the assumption that $\varphi(z), \psi(z) \in C^{(1,h)}(\mathcal{D} + \Gamma)$, was noted by N. E. Tovmasyan.

Along with problem (I)–(II), consider the homogeneous adjoint problem

$$\partial^2 v / \partial z^2 = 0; \tag{I*}$$

$$v|_\Gamma = 0. \tag{II*}$$

Starting from Green' s formula

$$\iint_{\mathcal{D}} \left[\bar{v} \frac{\partial^2 u}{\partial \bar{z}^2} - u \frac{\partial^2 \bar{v}}{\partial z^2} \right] dx\, dy = \frac{1}{2i} \int_\Gamma \left(\bar{v} \frac{\partial u}{\partial \bar{z}} - u \frac{\partial \bar{v}}{\partial z} \right) dz$$

for a solution $u(z)$ of problem (I)–(II) and for any solution $v(z)$ of the homogeneous adjoint problem (I*)–(II*)*, we have

$$\int_\Gamma f \frac{\partial \bar{v}}{\partial z} dz = 0. \tag{5}$$

Condition (5) is necessary for the solvability of problem (I)–(II). If condition (5) is also sufficient for the solvability of problem (I)–(II), then the latter problem is called **Hausdorff**. As a result of work [5] and Lemma 1 it follows

Theorem 2. For a simply connected domain bounded by a Lyapunov curve, problem (I)–(II) is Hausdorff if and only if $z(\omega)$ is a rational function.

From Theorems 1 and 2 it follows

Theorem 3. For a simply connected domain bounded by a Lyapunov curve, problem (I)–(II) is not Noetherian.

Let $\varphi_i(\omega)$ be the system of functions determined from (4), where $z(\omega)$ is the indicated conformal mapping, and let $\omega(z)$ be the function inverse to $z(\omega)$. Denote by $\Omega^+(s)$ the boundary value on Γ of any function of the class $E_2(\mathcal{D})$.

Theorem 4. In the case of the Hausdorff property of problem (I)–(II), for its solvability it is necessary and sufficient that the right-hand side $f(s)$ in condition II have the form

$$f(s) = \Omega^+(s) + \sum_{i=1}^{m} c_i \, \overline{\omega(s)} \, \overline{\varphi_i(\omega(s))},$$

where c_i are arbitrary constants.

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Note: Figure translations are in progress. See original paper for figures.