AN A PRIORI ESTIMATE FOR THE DERIVATIVE OF A SOLUTION OF A PARABOLIC EQUATION AND SOME OF ITS APPLICATIONS

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AN A PRIORI ESTIMATE FOR THE DERIVATIVE OF A SOLUTION OF A PARABOLIC EQUATION AND SOME OF ITS APPLICATIONS

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In the rectangle $Q\{(t,x) : 0 \le t \le T,\ 0 \le x \le l\}$ consider the parabolic equation

$$u_t = a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x) \tag{1}$$

under the following assumptions concerning the measurable functions $a(t,x,u,p)$ and $b(t,x,u,p)$: for $(t,x) \in Q$, $|u| \le M$, and arbitrary p,

$$0 < a_0 \le a(t,x,u,p) \le a_1(|p|^m + 1), \quad m \ge 0, \tag{2}$$

$$|b(t,x,u,p)| \le K a(t,x,u,p)(p^2+1). \tag{3}$$

Let the function $u(t,x)$ be continuous in Q and satisfy equation (1), with $|u(t,x)| \le M$. In the present paper a priori estimates are established for the modulus of the derivative u_x and for its Hölder constants, depending only on M, a_0, a_1, m, and K, i.e., without any assumptions whatsoever on the continuity of the coefficients of the equation. In the particular case of the linear equation

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u + d(t,x) \tag{4}$$

these estimates refine the known Schauder-type estimates (see ([1])): the Hölder norm of a bounded solution of equation (4) is estimated in ([2]).

The results obtained are applied to the study of boundary-value problems and the Cauchy problem for equation (1), as well as to a nonlinear parabolic equation of the form

$$u_t = a(t,x,u,u_x,u_{xx}). \tag{5}$$

The proofs of the a priori estimates are based on the fact that, by means of a certain device of introducing an additional spatial variable (see also ([3],[4])), the derivation of an interior estimate can be reduced to the study of the solution of a new quasilinear parabolic equation near a part of the boundary of a certain three-dimensional domain, where this solution vanishes; moreover, the smallness estimate for such a solution near the indicated part of the boundary does not depend on the smoothness of the coefficients of the equation (see, for example, Lemma 3 in ([5])).

1. A priori estimates. We introduce some notation. Let the function $u(t,x)$ be defined on some set D; for $\gamma \in (0,1)$ define the norms:

$$|u|_\gamma^D = \sup_{(t,x)\in D} |u(t,x)| + \sup_{\substack{(t,x)\in D \\ (\tau,y)\in D}} \frac{|u(t,x)-u(\tau,y)|}{\left(|t-\tau|^{1/2}+|x-y|\right)^\gamma},$$

$$|u|_{1+\gamma}^D = |u|_\gamma^D + |u_x|_\gamma^D.$$

By Q^δ and Q_δ we shall denote, respectively, the rectangles

$$\{(t,x) : 0<\delta\le t\le T,\ \delta\le x\le l-\delta\}$$

and

$$\{(t,x) : 0\le t\le T,\ \delta\le x\le l-\delta\}, \qquad \delta < l/2.$$

Let Γ be the lower base and the lateral sides of the rectangle Q. Any constants depending only on M, a_0, a_1, m, and K will be denoted by C; if a constant also depends on δ, then it is denoted by C_δ.

Theorem 1. Let the function $u(t,x)$ be continuous in Q and satisfy equation (1), for which conditions (2) and (3) are fulfilled; let $|u(t,x)|\le M$ in Q. Then

$$\sup_{Q^\delta} |u_x| \le C_\delta, \qquad |u|_{2/3}^{Q^{2\delta}} \le C_\delta.$$

If the function $u(t,x)$ has in Q generalized derivatives u_{xx} and u_{tx} summable with square, then there exists a $\gamma=\gamma(M,a_0,a_1,m,K,\delta)$ such that

$$|u|_{1+\gamma}^{Q^{2\delta}} \le C_\delta.$$

If, in addition, $u(0,x)\equiv 0$, then

$$|u|_{1+\gamma}^{Q_{2\delta}} \le C_\delta.$$

The following theorem, in whose proof Theorem 1 is used, concerns an estimate up to the boundary.

Theorem 2. Let the function $u(t,x)$ have in Q continuous derivatives u_t, u_x, u_{xx}, u_{tx} and satisfy equation (1), with conditions (2) and (3) fulfilled. Let $u(t,x)|_\Gamma = 0$ and $|u(t,x)| \le M$ in Q. Then, for some $\gamma = \gamma(M, a_0, a_1, m, K)$,

$$|u|^Q_{1+\gamma} \le C.$$

We note that the case of nonzero, but sufficiently smooth, boundary values of the function $u(t,x)$ on Γ is reduced to Theorem 2.

The main part of the proof of Theorem 1 is the derivation of an a priori estimate for the modulus of the derivative u_x, where the following lemma is used essentially; it is analogous to Lemma 3 in [5] and is of some independent interest (in this lemma $x = (x_1, \dots, x_n) \in R_n(x)$,

$$|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}, \quad v_x = (v_{x_1}, \dots, v_{x_n}), \quad G^\tau_r \text{ is the half-cylinder } \{(t,x) : -\tau < t \le 0,\ |x| \le r,\ x_1 > 0\}\,).$$

Lemma. Let the function $v(t,x)$ be continuous in $G^{2h}_{2\rho}$ and satisfy the equation

$$v_t - \sum_{i,j=1}^n a_{ij}(t,x,v_x) v_{x_i x_j} = b(t,x,v_x),$$

and suppose that for $(t,x) \in G^{2h}_{2\rho}$ the following conditions are fulfilled:

$$1) \quad \sum_{i,j=1}^n a_{ij}(t,x,v_x)\xi_i\xi_j \ge a_0 \sum_{i=1}^n \xi_i^2, \qquad a_0 = \text{const} > 0;$$

$$2) \quad \text{for some } \beta \in (0,1] \quad \sum_{i,j=1}^n |a_{ij}(t,x,v_x)| \le K_1[A(v)^{1-\beta} + 1],$$

where

$$A(v) = \sum_{i,j=1}^n a_{ij}(t,x,v_x) v_{x_i} v_{x_j};$$

$$3) \quad |b(t,x,v_x)| \le K_2[A(v) + 1].$$

Let $|v(t,x)| \le M$ in $G^{2h}_{2\rho}$ and $v|_{x_1=0} = 0$. Then for $(t,x) \in G^h_\rho$

$$|v(t,x)| \le \bar{N} x_1,$$

where $\bar{N}$ depends only on $M, a_0, K_1, K_2, \beta, \rho, h$.

In conclusion of this section, we explain the proof of Theorem 1 on the example of the linear equation (4), in which

$$0 < a_0 \le a(t,x) \le a_1, \qquad |b(t,x)| \le b_0, \quad |c(t,x)| \le c_0, \quad |d(t,x)| \le d_0.$$

We note that the function

$$v(t,x,y) \equiv u(t,x) - u(t,y)$$

in the prism

$$\{(t,x,y) : 0 < t \le T,\ 0 < y < x < l\}$$

satisfies the equation

$$v_t = a(t,x)v_{xx} + a(t,y)v_{yy} + \mu(t,x,y,v_x,v_y),$$

where

$$|\mu(t,x,y,v_x,v_y)| \le \mu_0(|v_x| + |v_y| + 2), \quad \mu_0 = 2\max(Mc_0, b_0, d_0).$$

Obviously, $|v| \le 2M$ and $v|_{x=y} = 0$. To the function $v(t,x,y)$ one may apply the lemma, according to which, for $\delta \le t \le T$ and $\delta \le y \le x \le l-\delta$, the inequality

$$|v(t,x,y)| = |u(t,x) - u(t,y)| \le C_\delta(x-y).$$

is valid. Consequently, $|u_x(t,x)| \le C_\delta$ for $(t,x) \in Q^\delta$; from this estimate it follows that

$$|u|_{2/3}^{Q^{2\delta}} \le C_\delta$$

(see Lemma 6 in ([5])).

In order to estimate the Hölder norm of the function $z = u_x$, assuming (for simplicity) that the solution $u(t,x)$ and the coefficients of equation (4) are sufficiently smooth, we use the known result of Nash ([6]) and its generalization in ([3]). Let $f(t,x) = b(t,x)u_x + c(t,x)u + d(t,x)$; since $|f(t,x)| \le f_0$ in Q^δ and $z_t = (az_x)_x + f_x$, the function $w = z + y$ satisfies the parabolic equation

$$w_t = (aw_x)_x + (fw_x)_y + (fw_y)_x + Bw_{yy},$$

where $B = 1 + (2f_0^2/a_0)$. As a consequence of the results of ([3], [6]),

$$|z|_\gamma^{Q^{2\delta}} \le C_\delta$$

for some $\gamma = \gamma(a_0, a_1, f_0) \in (0,1)$.

2. Boundary-value problems and the Cauchy problem. Numerous works ([7]–[9] and others) are devoted to the nonlocal theory of boundary-value problems and the Cauchy problem for equation (1); the technique for proving existence theorems for solutions of these problems in the presence of an a priori estimate for $|u|_{1+\gamma}^{Q}$ is well developed. Therefore, without going into details, we note only that Theorems 1 and 2 make it possible to study equation (1) with conditions (2) and (3) under very small assumptions on the coefficients a and b: it is enough to require that from equation (1) there follow an a priori estimate for the modulus of the solution of the problem under consideration and that the functions $a(t,x,u,p)$ and $b(t,x,u,p)$ satisfy a Hölder condition in any bounded domain with respect to all arguments (cf. the corresponding results for the linear equation (4)).

We now consider the nonlinear equation (5), assuming that the function $a(t,x,u,p,r)$ satisfies a Hölder condition in t and is continuously differentiable with respect to the remaining arguments, and that for $(t,x) \in Q$, $|u| \le M$, and arbitrary p and r,

$$0 < a_0 \le a_r(t,x,u,p,r) \le a_1(|p|^m + 1), \quad m \ge 0, \tag{6}$$

$$|a(t,x,u,p,0)| \le K_1(p^2+1)a_r(t,x,u,p,r), \tag{7}$$

and if, moreover, $|p| \le p_0$, then

$$|a_x| + |a_u| + |ra_p| \le K_2(r^2+1). \tag{8}$$

Theorem 3. *Suppose that for the function $a(t,x,u,p,r)$ inequalities (6)-(8) are satisfied and $a_u(t,x,u,0,0) \le a_2$ for $(t,x) \in Q$ and all u. Then there exists a solution $u(t,x)$ of equation (5), satisfying the condition $u|_\Gamma = 0$, and for some $\gamma > 0$*

$$|u|_{1+\gamma}^{Q} < \infty$$

and, for any δ,

$$|u_t|_{\gamma}^{Q_\delta} + |u_{xx}|_{\gamma}^{Q_\delta} < \infty. \tag{9}$$

Theorem 4. *If the conditions on the function $a(t,x,u,p,r)$ in Theorem 3 are satisfied for $(t,x) \in \Pi\{(t,x) : 0 \le t \le T,\ -\infty < x < +\infty\}$ and $|a(t,x,0,0,0)| \le a_3$, then there exists a unique solution $u(t,x)$ of the Cauchy problem for equation (5) with zero initial condition in the class of functions characterized by the inequality*

$$|u|_{1+\gamma}^{\Pi} + |u_t|_{\gamma}^{\Pi} + |u_{xx}|_{\gamma}^{\Pi} < \infty. \tag{10}$$

It follows from conditions (6) and (7) that in Theorems 3 and 4 equation (5) is almost quasilinear. However, by similar methods equation (5) is also studied in the case when the function $a(t, x, u, p, r)$ has arbitrary growth with respect to r. For example, the following holds.

Theorem 5. Let the function $u(t, x)$ be continuous in Q and satisfy equation (5), with $|u(t, x)| \leq M$; suppose there exist constants $a_0 > 0$, $K > 0$, and H such that, for $(t, x) \in Q$, $|u| \leq M$, and arbitrary real p and r, the inequalities

$$a_r(t, x, u, p, r) \geq a_0 > 0,$$

$$\pm a(t, x, u, p, \mp Kp^2) \leq H$$

hold.* Then for $(t, x) \in Q^\delta$ the estimate

$$|u_x(t, x)| \leq C(M, a_0, K, H, \delta)$$

is valid.**

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* These inequalities are, respectively, the parabolicity condition and the condition of subordination of the lower-order terms.
** This estimate is used in proofs of existence theorems for solutions of boundary-value problems and the Cauchy problem for equation (5).