MATHEMATICS

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1966

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Abstract

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MATHEMATICS

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ON INVARIANT SUBSPACES OF LINEAR OPERATORS ACTING IN A SPACE WITH AN INDEFINITE METRIC

(Presented by Academician L. S. Pontryagin on 5 XI 1965)

1. Let $\mathfrak{H}$ be a J-space ([1]), i.e., a Hilbert space in which, in addition to the usual scalar product (x, y), an indefinite scalar product

$$[x, y] = (Jx, y), \qquad J = P_+ - P_-,$$

is introduced, where P_+ and P_- are two mutually complementary orthogonal projectors. Put

$$\mathfrak{P}_+ = \{x : [x, x] \geq 0\};$$

$\mathfrak{P}_-$ and $\mathfrak{P}_0$ are defined analogously. By $\mathfrak{M}_1(\mathfrak{M}_2)$ we denote the set of all maximal subspaces from $\mathfrak{P}_+(\mathfrak{P}_-)$. Two subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are called J-orthogonal ($\mathfrak{L}_1 \perp \mathfrak{L}_2$) if $[\mathfrak{L}_1, \mathfrak{L}_2] = \{0\}$. For any set $\mathfrak{G} \subset \mathfrak{H}$ put

$$\mathfrak{G}^{\perp} = \{x : [x, \mathfrak{G}] = \{0\}\}.$$

Denote by $\mathfrak{R}$ the ring of all linear bounded operators A acting in $\mathfrak{H}$, and by $\mathfrak{S}_\infty$ its ideal consisting of all completely continuous $A \in \mathfrak{R}$.

For arbitrary $A \in \mathfrak{R}$, the equality

$$[Ax, y] = [x, A^+ y] \qquad (x, y \in \mathfrak{H})$$

defines the J-adjoint A^+; if $A = A^+$ ($U^+U = UU^+ = I$), then the operator A (U) is called J-selfadjoint (J-unitary).

We shall say that a commutative (with respect to multiplication) family $\mathfrak{A} \subset \mathfrak{R}$ has property (P) if, for each pair of subspaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfying the conditions

$$1)\ \mathfrak{L}_1 \subset \mathfrak{P}_+,\ \mathfrak{L}_2 \subset \mathfrak{P}_-; \qquad 2)\ \mathfrak{L}_1 \perp \mathfrak{L}_2; \qquad 3)\ A\mathfrak{L}_j \subset \mathfrak{L}_j\ (j = 1, 2;\ A \in \mathfrak{A}),$$

there exists such a pair of subspaces $\mathfrak{L}_1^{\max}$ and $\mathfrak{L}_2^{\max}$ that

$$1')\ \mathfrak{L}_j \subset \mathfrak{L}_j^{\max} \in \mathfrak{M}_j; \qquad 2')\ \mathfrak{L}_1^{\max} \perp \mathfrak{L}_2^{\max}; \qquad 3')\ A\mathfrak{L}_j^{\max} \subset \mathfrak{L}_j^{\max}\ (j = 1, 2;\ A \in \mathfrak{A}).$$

If the family $\mathfrak{A}$ consists of one operator and has property (P), then we shall say that this operator has property (P).

R. S. Phillips ([2]) conjectured that an arbitrary commutative algebra $\mathfrak{A} \subset \mathfrak{R}$, closed with respect to the operation of J-adjunction ($\mathfrak{A} = \mathfrak{A}^+$), has property (P), and proved it for the case when $\mathfrak{A} = \mathfrak{A}^+ - \mathfrak{A}^*$. It is unknown whether an arbitrary J-selfadjoint (J-s.a.) or J-unitary operator has property (P). In this connection, the following generalization of Theorem 3 from ([3]) is of interest.

Theorem 1. *Every J-unitary operator U for which*

$$P_+ U P_- \in \mathfrak{S}_\infty$$

has property (P).

We give some explanations for the proof of the theorem. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be subspaces with properties 1)–3). Without loss of generality one may assume that

$$U\mathfrak{L}_j = \mathfrak{L}_j \qquad (j = 1, 2).$$

Represent $\mathfrak{L}_j$ in the form

$$\mathfrak{L}_j = \{P_j(x) + K_{jP_{jx}} : x \in \mathfrak{H}_j\},$$

where $\mathfrak{H}_1 = P_+\mathfrak{H}$, $\mathfrak{H}_2 = P_-\mathfrak{H}$; P_j is the orthoprojector in $\mathfrak{H}_j$; $K_j \mid \mathfrak{H}_j \to \mathfrak{H}_k$, $\|K_j\| \leq 1$, $j, k \neq 1, 2$, $j \neq k$*. Then the following holds.

Lemma 1. *A subspace $\mathfrak{L} \in \mathfrak{M}_1$ satisfies the conditions $\mathfrak{L} \supset \mathfrak{L}_1$, $\mathfrak{L} \perp \mathfrak{L}_2$ if and only if for its angular operator $K_{\mathfrak{L}}$ the equality*

$$K_{\mathfrak{L}} = K_1P_1 + (K_2P_2)^*(I - P_1) + (I - P_2)K_{\mathfrak{L}}(I - P_1)$$

holds.

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* For $\mathfrak{L}_j \in \mathfrak{M}_j$, the operator K_j coincides with the angular operator ([1]) of the subspace $\mathfrak{L}_j$ $(j = 1, 2)$.

Further, Theorem 1 is proved by the same method as in ([3], [4]); one need only consider, instead of the set $\mathfrak{K}_+$ from ([3]), the set of all bounded linear operators X acting from $(I - P_1)\mathfrak{H}_1$ into $(I - P_2)\mathfrak{H}_2$, such that

$$\|K_1P_1 + (K_2P_2)^*(I - P_1) + (I - P_2)X(I - P_1)\| \leq 1$$

and apply the fixed-point theorem 1 from ([4]).

Similarly one proves

Theorem 2. *Let the operator $A \in \mathfrak{R}$ satisfy the conditions $A\mathfrak{P}_+ \subset \mathfrak{P}_+$, $P_+AP_- \in \mathfrak{S}_\infty$, and let $\mathfrak{L}$ be a subspace with the properties $\mathfrak{L} \subset \mathfrak{P}_+$, $A\mathfrak{L} = \mathfrak{L}$. Then there exists $\mathfrak{L}^{\max} \in \mathfrak{M}_1$ such that $\mathfrak{L}^{\max} \supset \mathfrak{L}$ and $A\mathfrak{L}^{\max} \subset \mathfrak{L}^{\max}$.*

2. We shall call an operator $B \in \mathfrak{R}$ J-definite if either $[Bx, x] \geq 0$ $(x \in \mathfrak{H})$, or $[Bx, x] \leq 0$ $[x \in \mathfrak{H}]$. We shall call a J-s.a. operator A J-definizable if there exists a nontrivial polynomial $p_A(\lambda)$ such that the operator $p_A(A)$ is J-definite. It is known, for example, that in the space Π_κ ([1]) every J-s.a. operator is J-definizable ([5]).

Theorem 3. *Every finite commutative family $\mathfrak{A}$ of J-definizable operators has at least one common invariant subspace $\mathfrak{L}^{\max} \in \mathfrak{M}_1$ $(A\mathfrak{L}^{\max} \subset \mathfrak{L}^{\max};\ A \in \mathfrak{A})$.*

As M. G. Krein kindly pointed out to me, from this theorem one easily obtains

Theorem 4. *Every commutative family $\mathfrak{A}$ of J-definizable operators A with the property $P_+AP_- \in \mathfrak{S}_\infty$ $(A \in \mathfrak{A})$ has at least one common invariant subspace $\mathfrak{L}^{\max} \in \mathfrak{M}_1$.*

Theorem 4 contains the theorem of M. A. Naimark from ([6], [7]).

An operator $F \in \mathfrak{R}$ with the property $F^2 = F = F^+$ is called a J-projector; two J-projectors F, G are called J-orthogonal if $FG = 0$. Theorem 3 follows from the following more general proposition.

Theorem 5. *Let $\mathfrak{A}$ be a commutative family of J-s.a. operators, $\mathfrak{F}$ a family of mutually J-orthogonal J-projectors for which $AF = FA$ $(A \in \mathfrak{A},\ F \in \mathfrak{F})$, and suppose that for every pair of operators A, F $(A \in \mathfrak{A},\ F \in \mathfrak{F})$ there exists a natural number $n = n(A, F)$ such that the operator $A^{n}F$ is J-definite. Then there exists $\mathfrak{L}^{\max} \in \mathfrak{M}_1$, invariant with respect to all operators F and AF $(F \in \mathfrak{F},\ A \in \mathfrak{A})$.*

In the proof of Theorems 3 and 5 it is essential that a J-definizable operator possesses "its own spectral function with critical points," analogous to that considered in ([5]) for J-s.a. operators in Π_κ.* The following two lemmas are also used; they are of independent interest.

Lemma 2. *Every family $\mathfrak{F}$ of mutually J-orthogonal J-projectors has property (P).*

In the case when the family $\mathfrak{F}$ consists only of the identity operator, the assertion of Lemma 2 follows from the result of Phillips formulated above, which is also used in the proof of Lemma 2.

Lemma 3. *Let $\mathfrak{A}$ be a commutative family of J-s.a. operators. Then there exists a subspace $\mathfrak{N}_0 \subset \mathfrak{H}$ satisfying the conditions: $\mathfrak{N}_0 \subset \mathfrak{P}_0$, $A\mathfrak{N}_0 \subset \mathfrak{N}_0$, and $A\mathfrak{N}_0^\perp \cap (A\mathfrak{N}_0^\perp)^\perp \subset \mathfrak{N}_0$ $(A \in \mathfrak{A})$.*

Theorem 6. *Every commutative family $\mathfrak{A}$ of J-definite operators has property (P).*

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* We take this opportunity to point out, on behalf of the authors of note ([5]), that an error was made in writing formula (2) of that note. The correct form of this formula, in the notation of the note, is as follows:

$$\mathcal{P}_{A^2}(A)E(\Delta)x = SE(\Delta)x + \int_\Delta \mathcal{P}_{A^2}(\lambda)\,dE_\lambda x \qquad (x \in \Pi_\kappa),$$

where S is a s.a. operator with the properties: 1) $S^2 = 0$; 2) $(Sx, x) \geq 0$ $(x \in \Pi_\kappa)$, and 3) $SE(\Delta') = 0$ for every segment Δ' containing no critical point.

The assertions of Theorems 3 and 6 are apparently new even for the case when the family $\mathfrak{A}$ consists of only one operator.

3. Let us now consider the pencil $L(\lambda) = \lambda^2 I + \lambda B + C$, where B and C are two bounded self-adjoint operators in the Hilbert space $\mathfrak{K}$. The pencil $L(\lambda)$ is called **relatively strongly damped** (r.s.d.) ([8]) if

$$(By, y)^2 > 4(Cy, y)\|y\|^2 \qquad (y \in \mathfrak{K};\ y \neq 0) \tag{*}$$

and **strongly damped** (s.d.) if, in addition, $B, C \geq 0$. Condition $(*)$ means that each quadratic trinomial

$$\varphi_y(\lambda) = (L(\lambda)y, y) \qquad (y \in \mathfrak{K},\ y \neq 0)$$

has two distinct real roots $p_{1,2}(y)$, $p_2(y) < p_1(y)$. Put $\alpha_1 = \inf p_1(y)$, $\alpha_2 = \sup p_2(y)$ $(y \neq 0)$. It turns out (cf. ([9])) that $-\infty < \alpha_2 \leq \alpha_1 < \infty$. We shall say that a point $\lambda \in \sigma(L)$ ([9]) of the r.s.d. pencil $L(\lambda)$ belongs to $\sigma_1(L)$ $(\sigma_2(L))$, if for every sequence (y_n) for which $\|y_n\| = 1$ and $L(\lambda)y_n \to 0$, one has $p_1(y_n) \to \lambda$ $(p_2(y_n) \to \lambda)$. It can be shown that

$$\sigma(L) \cap [\alpha_1, \infty) = \sigma_1(L) \cup \{\alpha_1\}, \qquad \sigma(L) \cap (-\infty, \alpha_2] = \sigma_2(L) \cup \{\alpha_2\}.$$

As in ([9,10]), with the s.d. pencil $L(\lambda)$ we associate the J-s.s. operator

$$H = \begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & -B \end{pmatrix} \quad \text{in the } J\text{-space } \mathfrak{H} = \mathfrak{K} \oplus \mathfrak{K}, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We shall say ([11]) that a J-s.s. operator A in the J-space $\mathfrak{H}$ satisfies the Pesonen condition if from the equalities $[x, x] = 0$ and $[Ax, x] = 0$ it follows that $x = 0$.

Lemma 4. *The pencil $L(\lambda)$ is s.d. if and only if the operator H satisfies the Pesonen condition.*

Theorem 7. *Let the pencil $L(\lambda)$ be r.s.d. Then there exist two roots Z_1 and $Z_2 = -B - Z_1^*$ of the equation $Z^2 + BZ + C = 0$ such that:*

1) $\sigma(Z_j) = \sigma_j(L) \cup \{\alpha_j\}$, and the eigenvectors ([9]) of the pencil $L(\lambda)$ and of the root Z_j, corresponding to eigenvalues from $\sigma_j(L) \setminus \{\alpha_k\}$, coincide $(j, k = 1, 2;\ j \neq k)$;

2) the roots Z_1 and Z_2 are symmetrized by the positive operator

$$S = Z_1 + Z_1^* + B = Z_1 - Z_2;$$

3) the roots Z_1, Z_2 form a complete pair ([9,10]);

4) for each segment Δ of the real axis lying to the right of α_2 (to the left of α_1), there exist two subspaces $\mathfrak{K}_\Delta, \mathfrak{K}'_\Delta$, invariant with respect to the operator Z_1 (Z_2), such that $\mathfrak{K} = \mathfrak{K}_\Delta \dotplus \mathfrak{K}'_\Delta$ and $\sigma(Z_1|\mathfrak{K}_\Delta) \subset \Delta$, $\sigma(Z_1|\mathfrak{K}'_\Delta) \subset \Delta'$ ($\sigma(Z_2|\mathfrak{K}_\Delta) \subset \Delta$, $\sigma(Z_2|\mathfrak{K}'_\Delta) \subset \Delta'$), where $\Delta' = (-\infty, \infty)\backslash\Delta$. The operator Z_1 (Z_2) in the subspace $\mathfrak{K}_\Delta$ is similar to a self-adjoint operator. If the pencil $L(\lambda)$ is strongly damped, then, in addition, the inequalities

$$Z_1^* Z_1 \leq C, \qquad Z_2^* Z_2 > C$$

hold.

To the proof of Theorem 7 we add the following explanations. By the change of parameter $\lambda \to \lambda - b$ (b a sufficiently large real number), an r.s.d. pencil can be reduced to an s.d. pencil $L'(\lambda) = \lambda^2 I + \lambda B' + C'$, where $B' = 2bI + B$, $C' = b^2 I + bB + C$ ([8]). Therefore it is enough to consider the case of an s.d. pencil. For such a pencil the operator H satisfies the Pesonen condition (Lemma 4), by virtue of which, by a lemma of R. Cohn ([11]), there will exist a real μ such that the operator $H - \mu I$ is J-definite ($\alpha_2 \leq \mu \leq \alpha_1$). Then, by Theorem 3, the operator H will have a maximal

a nonnegative invariant subspace. After this, the root Z_1 is obtained in the same way as in ([9], [10]). Theorem 4 is proved with the aid of the above-mentioned "proper spectral function" of the J-definitizable operator H.

Theorem 7, in comparison with the corresponding theorem from ([9], [10]), is more general in the sense that it does not assume complete continuity of the operator C.

Finally, we note that Theorem 7 can also be extended to the case of an unbounded operator B.

I express my sincere gratitude to Prof. M. G. Krein for discussion and for very valuable suggestions.

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Received
3 XI 1965

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Note: Figure translations are in progress. See original paper for figures.