DISTRIBUTION OF NONRIGID SLEEVE CONSTRAINTS FOR A CONVEX SURFACE

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Abstract

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DISTRIBUTION OF NONRIGID SLEEVE CONSTRAINTS FOR A CONVEX SURFACE

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The present work is devoted to infinitesimal bendings of convex surfaces under sleeve constraints (1). We consider nonsingular sleeves, i.e., sleeves nowhere at the boundary nonorthogonal to the surface. A sleeve constraint is called **nonrigid** if the surface subject to the sleeve constraint and fixed at some point together with the tangent plane admits nontrivial infinitesimal bendings; otherwise the sleeve is called rigid. The aim of the work is to study the distribution of nonrigid sleeve constraints among all nonsingular ones. It is established that, for surfaces with an arbitrary smooth boundary, the nonrigid sleeves form a closed set. For a narrower class of surfaces—convex surfaces bounded by a circle —a more complete picture of the distribution of nonrigid sleeves is given. It is shown that every sleeve can be included in a one-parameter family of sleeve constraints, among which the nonrigid ones form at most a countable set; if the latter are countable, then they converge to the orthogonal sleeve. In some cases it is possible to prove the existence of a countable set of nonrigid sleeves.

Related questions are treated in § 11 of Chapter 5 of I. N. Vekua' s book (1).

I. Consider a simply connected surface of Gaussian curvature strictly positive up to the boundary. Suppose that the surface belongs to the class $D_{3,p}$, $p > 2$, and that the boundary of the surface is a simple closed curve of class C^1_μ, $0 < \mu < 1$.

Theorem 1. *If a sequence of nonrigid sleeve constraints Σ_n of the surface converges to a nonsingular sleeve Σ, then the sleeve constraint Σ is nonrigid.*

Remark. If the sleeve Σ is singular, then the assertion of the theorem is, generally speaking, false.

Proof. We shall show that rigid nonsingular sleeves constitute an open set, i.e., if a sleeve $\widetilde{\Sigma}$ is rigid, then all sleeves sufficiently close (to $\widetilde{\Sigma}$) in a certain sense are also rigid. For this purpose consider, in a plane domain D with boundary Γ, the equations of infinitesimal bendings of the surface S with the boundary condition generated by the sleeve constraint (see, for example, (1), p. 467):

$$\partial_{\bar{z}} w + Aw + B\overline{w} = 0 \quad \text{in } \overline{D};$$

$$\operatorname{Re}\{z^2\partial_z w + [a(z) + b(z)\operatorname{tg}\varphi]w\} = 0 \quad \text{on } \Gamma. \tag{1}$$

Here w is the unknown function; A, B, $a(z)$, $b(z)$ are given sufficiently smooth functions; φ is the angle between the surface and the sleeve. Without loss of generality, the domain D may be regarded as the unit disk. Let the sleeve $\widetilde{\Sigma}$ be rigid; then problem (1) with $\varphi = \widetilde{\varphi}(s)$ has only the zero solution. But then, according to the results of (^{4}), the nonhomogeneous problem

$$\partial_{\bar{z}} w + Aw + B\overline{w} = 0 \quad \text{in } D;$$

$$\operatorname{Re}\{z^2\partial_z w + [a(z) + b(z)\operatorname{tg}\widetilde{\varphi}]w\} = c(s) \quad \text{on } \Gamma; \tag{2}$$

where $c(s)$ is a given function, solvable uniquely when a finite number of solvability conditions are satisfied. If a solution exists, it is given by the formula $w = A(c)$, where A is a certain linear operator acting on Γ in some C_α, $0 < \alpha < 1$. Thus every solution of problem (1) satisfies the equation

$$w + A[b(z)(\operatorname{tg}\varphi - \operatorname{tg}\varphi)w] = 0.$$

The latter has only the zero solution if φ, in the norm $C_\alpha(\Gamma)$, $0 < \alpha < 1$, differs little from φ. The theorem is proved.

II. Let a surface S, located in the half-space $z \geq z_0 > 0$ and convex downward, be given in the rectangular Cartesian coordinate system $Oxyz$ by the equation $z = f(x, y)$. We shall assume that the Gaussian curvature $K \geq k_0 > 0$; that the regularity of the surface is sufficiently good, for example, $f(x, y) \in C^4_\alpha$, $0 < \alpha < 1$, and that in a neighborhood of the origin $f_{xx} = f_{yy}$, $f_{xy} = 0$. Suppose further that the surface is bounded by a circle L, projecting onto the plane Oxy as the circle $\Gamma : x^2 + y^2 = 1$. We denote the domain $x^2 + y^2 \leq 1$ by D.

Let the surface S be subjected to an oblique collar constraint Σ. Denote by $\mathbf{n}_\Sigma$ the normal to Σ along L. The orientation of the normal $\mathbf{n}_\Sigma$ is chosen arbitrarily at some point of the edge L, and then is extended by continuity to the entire contour L. Next consider the orthogonal collar of the surface S. We choose the normal $\mathbf{n}$ on it in an analogous way. Denote by α the angle between $\mathbf{n}$ and $\mathbf{n}_\Sigma$. The angle α is measured from $\mathbf{n}$ to $\mathbf{n}_\Sigma$ counterclockwise, when viewed from the side of the positive direction of the curve L. Obviously, $\alpha = \alpha(s)$, where s is the arc length of the contour L. Without loss of generality, we assume $0 < \alpha < \pi$.

Under the assumptions made concerning the surface, the following theorem holds.

Theorem 2. *Let a collar constraint Σ_1, determined by an angle $\alpha_1(s)$, be given. Then among the collar constraints Σ_ε, determined by the angle*

$$\operatorname{tg}\alpha(s) = \frac{1}{\varepsilon}\operatorname{tg}\alpha_1(s),$$

where ε is a parameter, $\varepsilon \in (0, \infty)$, there exists at most a countable set of non-rigid constraints. If the number of the latter is finite, then they converge to the orthogonal collar.

We give the main stages of the proof of this theorem.

1. Fix the point $(0, 0, f(0, 0))$ on the surface together with the tangent plane and subject the surface S to an infinitesimal bending. Denote the components of the vector $\mathbf{U}$ of the displacement field of points of the surface by $\{\xi, \eta, \zeta\}$. Then, as is known (^{1}), the latter satisfy the system of differential equations of elliptic type:

   $$\lambda_x = r\zeta; \qquad \lambda_y + \mu_x = 2s\zeta; \qquad \mu_y = t\zeta,$$

   where $\lambda = \xi + p_x\zeta$, $\mu = \eta + f_y\zeta$, $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$. Putting further $w(z) = \lambda + i\mu$, $z = x + iy$ ($i^2 = -1$), this system can be written in complex form. Using the fact that, in a neighborhood of the origin, $r = t$, $s = 0$, and the point $(0, 0, f(0, 0))$ is fixed on the surface, the function $w(z)$ can be represented in the form $w(z) = zw_1(z)$, where $w_1(z)$ belongs to the class C^2_α, $0 < \alpha < 1$ (under the assumptions made, $w_1(z)$ is even analytic in a neighborhood of $z = 0$). Denoting $\operatorname{Re} w_1(z) = U$, we obtain, for finding $U(x, y)$, a differential equation of elliptic type

   $$WU \equiv \sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial U}{\partial x_i}\right) + \sum_{i=1}^{2} e_i \frac{\partial U}{\partial x_i} + cU = 0, \tag{1'}$$

   where $x_1 = x$, $x_2 = y$; $a_{ik} \in C^2_\alpha(\overline{D})$; e_i, $c \in C^1_\alpha(\overline{D})$; $\alpha < 1$; $a_{ik} = a_{ki}$.

2. Subject the surface S to the collar constraint Σ. This means that the displacement of points of S along L lies in the planes tangent to Σ, i.e. $\mathbf{U}\mathbf{n}_\Sigma = 0$ on L. Without loss of generality, one may assume that the vector $\mathbf{n}_\Sigma$ along L has coordinates $\{\sin\gamma \cdot x; \sin\gamma \cdot y; \cos\gamma\}$, where $\gamma = \gamma(s)$ is the angle between Σ and the half-direction of the axis Oz. $\gamma > 0$ or $\gamma < 0$ according as the projection of the vector $\mathbf{n}_\Sigma$ onto the plane Oxy has the direction of the vector $\{x, y\}$ or the opposite direction. Then the boundary condition for the function $U(x, y)$ along L can be written in the form

$$PU \equiv a^{(l)}\frac{\partial U}{\partial l} + \beta U = 0, \tag{2'}$$

where the derivative is taken in the direction $\{x, y\}$, and the coefficients have the form

$$a^{(l)} = \frac{r\dot{x}^2 + 2s\dot{x}\dot{y} + t\dot{y}^2}{rx^2 + 2sxy + ty^2};$$

$$\beta = \frac{r\dot{x}^2 + 2s\dot{x}\dot{y} + t\dot{y}^2}{rx^2 + 2sxy + ty^2} + \frac{\sin\gamma\,(r\dot{x}^2 + 2s\dot{x}\dot{y} + t\dot{y}^2)}{\cos\gamma - f_l'\sin\gamma}. \tag{3}$$

Thus, the study of infinitesimal bendings of the surfaces under consideration with bushing constraints is reduced to the study of the solvability of the boundary-value problem (1′), (2′).

3. A conclusion about the solvability of problem (1′), (2′) can be drawn if the solvability of the adjoint problem is known. In the present case the adjoint problem turns out to be simpler than the original one. Namely, using the explicit form of the coefficients of equation (1), the equation adjoint to it can be written as follows:

$$RV \equiv \sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial V}{\partial x_i}\right) - \sum_{i=1}^{2} e_i^{\bullet}\frac{\partial V}{\partial x_i} = 0, \tag{4}$$

where V is the unknown function. It is important to note that equation (4) contains only derivatives of the unknown function.

The condition adjoint to condition (2) has the form (see, for example, (^{2}))

$$QV \equiv a^{(\lambda)}\frac{\partial V}{\partial \lambda} + (\beta - b - b')V = 0, \tag{5}$$

where λ is the direction adjoint to the direction l; $a^{(\lambda)} > 0$. The coefficient $(\beta - b - b')$ is determined by the formula

$$\beta - b - b' = 1 + \frac{r\dot{x}^2 + 2s\dot{x}\dot{y} + t\dot{y}^2}{\cos\gamma - f_l'\sin\gamma}\sin\gamma.$$

Taking into account that for the surfaces under consideration $r\dot{x}^2 + 2s\dot{x}\dot{y} + t\dot{y}^2 = f_l'$, this expression can be rewritten in the form

$$\beta - b - b' = \frac{\operatorname{ctg}\gamma}{\operatorname{ctg}\gamma - f_l'}.$$

Since $\gamma = \pi/2 - \alpha - \operatorname{arctg} f_l'$, we have $\operatorname{ctg}\gamma = (f_l' + \operatorname{tg}\alpha)/(1 - f_l'\operatorname{tg}\alpha)$, and therefore

$$\beta - b - b' = \frac{1}{1 + f_l'^2} + \frac{f_l'}{1 + f_l'^2}\operatorname{ctg}\alpha \equiv a^2(s) + b^2(s)\operatorname{ctg}\alpha.$$

4. The boundary-value problem

$$RV = 0 \text{ in } D; \qquad a^{(\lambda)}\frac{\partial V}{\partial \lambda} + a^2(s)V = f(s) \text{ on } \Gamma,$$

as is known (^{2}), has a unique solution for any function $f(s)$ of class $C^1_\alpha(\Gamma)$, $0 < \alpha < 1$. The solution belongs to the class $C^2_\alpha(\overline{D})$, $0 < \alpha < 1$, and is given by the formula $V = A(f)$, moreover $\|A(f)\|_{2,\alpha} \leq \text{const} \cdot \|f\|_{1,\alpha}$, where the constant does not depend on the function f (^{3}). It follows that problem (4), (5) is equivalent to the equation

$$V + A[b^2(s)\operatorname{ctg}\alpha\, V] = 0.$$

Let us now consider the one-parameter family of hinge connections determined by the angle

$$\operatorname{ctg}\alpha(s) = \varepsilon \operatorname{ctg}\alpha_1(s),$$

where ε is a parameter, $\varepsilon \in (0, \infty)$; $\alpha_1(s)$ is a prescribed function determining the hinge Σ_1. Then the search for nonrigid hinges reduces to finding nontrivial solutions of the equation

$$V + \varepsilon A_1(V) = 0, \tag{6}$$

where $A_1(V) \equiv A[b^2(s)\operatorname{ctg}\alpha_1(s)V]$. Since the set $\|V'\|_{2,\alpha} \leq \text{const}$ is compact in the Banach space C^1_α, $0 < \alpha < 1$, it follows from this that A_1 is a completely continuous mapping of C^1_α into C^1_α. Applying the Riesz theory of completely continuous operators to equation (6), we obtain the required result.

5. Application of Darboux' s theorem on the relation between projective transformations and infinitesimal bendings of surfaces makes it possible to establish the validity of Theorem 2 for surfaces bounded by a circle and arranged starwise with respect to some point.

III. **Theorem 3.** *Let a spherical segment be subject to a hinge connection Σ_1. Then, among the hinge connections Σ_ε indicated in Theorem 2, there exists an exactly countable set of nonrigid hinges converging to the orthogonal hinge.*

The assertion of the theorem follows from the fact that the operator in equation (7) for a spherical segment is self-adjoint.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.