ON ELEMENTARY STRUCTURAL PROPERTIES OF SEMIGROUPS

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ON ELEMENTARY STRUCTURAL PROPERTIES OF SEMIGROUPS

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In this note the following notation is adopted: $\Sigma'(\Gamma)$ is the structure of all subsemigroups (including the empty subsemigroup) of the semigroup Γ; $S(\mathfrak{A})$ is the class of structures isomorphic to the structures $\Sigma'(\Gamma)$ for all possible Γ belonging to the class of semigroups $\mathfrak{A}$. Strong bonds of one-element semigroups (see ([1])) will be called here simply **strong bonds**. For definitions of the notions from the theory of classes of models used here, see, for example, the survey of A. I. Mal' tsev ([2]).

One of the most natural refinements of the concept of structural characterizability (see ([3,4])), which plays an important role in the study of structural properties of semigroups, appears to be one in which the required conditions for the structure of subsemigroups must be formulated in the language of the restricted predicate calculus. We thus arrive at the concept of structural axiomatizability and other related notions*. Let $\mathfrak{A}$ be a subclass of the class of semigroups $\mathfrak{B}$ (here and everywhere below abstract classes are considered, i.e. classes closed under isomorphisms). We shall call the class $\mathfrak{A}$ **structurally axiomatizable in the class** $\mathfrak{B}$ if $S(\mathfrak{A})$ is relatively axiomatizable in the class $S(\mathfrak{B})$, i.e. if there exists an axiomatizable class of structures K such that
$S(\mathfrak{A}) = K \cap S(\mathfrak{B})$. Structural finite axiomatizability, etc., are defined analogously. We shall say that the class $\mathfrak{A}$ is **elementarily structurally definable in the class** $\mathfrak{B}$ if there exists an arithmetically closed class of structures K_1 such that
$S(\mathfrak{A}) = K_1 \cap S(\mathfrak{B})$. In the important case when $\mathfrak{B}$ is the class of all semigroups, the words "in the class $\mathfrak{B}$" in the corresponding definitions will be omitted.

By $\mathfrak{B}$ below we shall everywhere denote the class of all semigroups.

It is clear that from the structural axiomatizability of the class $\mathfrak{A}$ follows its elementary structural definability, while the latter implies structural definability in the sense of ([3,4]). It is also obvious that, in particular, the class $\mathfrak{A}$ will be structurally axiomatizable if the class $S(\mathfrak{A})$ is axiomatizable; $\mathfrak{A}$ will be elementarily structurally definable if the class $S(\mathfrak{A})$ is arithmetically closed. It can be shown that the listed properties of classes of semigroups are, generally speaking, distinct.

Examples of structurally axiomatizable classes of semigroups $\mathfrak{A}$ for which the corresponding classes $S(\mathfrak{A})$ are not arithmetically closed are easy to give if one uses the following theorem, which is not difficult to prove by applying the theorem on extensions of models (see (^{2})).

Theorem 1. *Let $\mathfrak{A}$ be an arbitrary class of semigroups containing an infinite semigroup that can be included in such a structurally axiomatizable class $\mathfrak{C}$ that the cardinalities of all semigroups from $\mathfrak{C}$ are bounded in the aggregate by some cardinal number. Then the class $S(\mathfrak{A})$ is not arithmetically closed.*

* Let us note at once that the basic definitions formulated in the note directly for semigroups apply essentially to the structures of subalgebras of arbitrary abstract algebras.

We note that this theorem is also valid for any classes of abstract algebras in which the infinitude of an algebra is equivalent to the infinitude of the structure of its subalgebras.

Let us indicate a number of special cases of Theorem 1.

Corollary 1. *If a class of semigroups $\mathfrak{A}$ contains an infinite cyclic semigroup, then the class $S(\mathfrak{A})$ is not arithmetically closed.*

Corollary 2. *If a class of semigroups $\mathfrak{A}$ contains some group of type p^∞, then the class $S(\mathfrak{A})$ is not arithmetically closed.*

Corollary 3. *If a class of semigroups $\mathfrak{A}$ contains an infinite cyclic group, then the class $S(\mathfrak{A})$ is not arithmetically closed.*

In particular, let us note the arithmetic non-closedness of the class $S(\mathfrak{B})$.

On the other hand, there exist classes of semigroups that are not structurally axiomatizable, which are elementarily structurally definable and, moreover, for which the corresponding classes of structures of subsemigroups are arithmetically closed. Examples of this kind are supplied, in particular, by the following theorem, if one takes in it, as $\mathfrak{A}$, a structurally definable class of finite semigroups (then it is not hard to verify that $S(\mathfrak{A})$ will be arithmetically closed), for example, the class of all finite semigroups, the class of finite semigroups of idempotents, etc.

Theorem 2. *If $\mathfrak{A}$ is an arbitrary class of semigroups containing all finite strongly connected components and, at least for one infinite cardinality, not containing strongly connected components of this cardinality, then $\mathfrak{A}$ is not structurally axiomatizable.*

The proof of Theorem 2 relies essentially on the fact that the structures of all subsets of arbitrary sets belong to the class $S(\mathfrak{B})$ (if Γ is a strongly connected component, then $\Sigma'(\Gamma)$ is the structure of all subsets of Γ), which makes it possible to apply A. D. Taimanov' s criterion (^{5}), Theorem 8; see also (^{2}), Theorem 3 from § 2.4), if one uses the following lemma.

Lemma 1. *Let* Σ *be the structure of all subsets of an infinite set. For any tuple* $(n_1, \dots, n_l)$ *there exists such a finite set that* $\Sigma(n_1, \dots, n_l)$ *is embeddable in the structure of all subsets of this set.*

By arguments analogous to those used for the proof of Lemma 1, one can show that the structures of all subsets of any two infinite sets are elementarily equivalent (this fact also follows from results of Yu. L. Ershov (^{6})). Hence there follows an example of a structurally definable class of semigroups $\mathfrak{A}$ that is not elementarily structurally definable: as $\mathfrak{A}$ one may take the class of strongly connected components of one and the same infinite cardinality.

All known classes of semigroups $\mathfrak{A}$ for which the corresponding classes $S(\mathfrak{A})$ are arithmetically closed consist of finite semigroups. The question arises: will it be true that, for any class of semigroups $\mathfrak{A}$ containing an infinite semigroup, the class $S(\mathfrak{A})$ is not arithmetically closed? The question as formulated is obviously equivalent to the following: for every infinite semigroup Γ, does there exist a structure elementarily equivalent to the structure $\Sigma'(\Gamma)$ and not belonging to the class $S(\mathfrak{B})$*? In the general case the author does not know the answer to this question. But, for example, for semigroups of idempotents this question is resolved positively. Namely, the following is true.

Theorem 3. *If a class of semigroups* $\mathfrak{A}$ *contains an infinite semigroup of idempotents, then the class* $S(\mathfrak{A})$ *is not arithmetically closed.*

Theorem 3, by the Löwenheim–Skolem theorem, follows from the fact that the class of semigroups of idempotents is structurally axiomatizable (see below), and from the following lemma.

* If the question were resolved positively, the following theorem would be true, describing all cases of arithmetic closedness of classes of structures of subsemigroups: *for a class of semigroups* $\mathfrak{A}$ *the class* $S(\mathfrak{A})$ *is arithmetically closed if and only if* $\mathfrak{A}$ *is a structurally definable class of finite semigroups.*

Lemma 2. If Γ is an infinite semigroup of idempotents, then $\Sigma'(\Gamma)$ is uncountable.

An analysis of the structural characteristics of classes of semigroups obtained in works ($^{3,4,7-9}$) shows that almost all of them either are elementary, or are easily reducible to elementary ones. In particular, let us note the structural finite axiomatizability of the following classes of semigroups: semigroups without idempotents, nonperiodic semigroups, periodic semigroups, periodic semigroups with one idempotent, the infinite cyclic semigroup, groups of type p^∞, semigroups of idempotents, rectangular nonsingular semigroups (see (^{9})), finite cyclic semigroups that are not improper groups. Further, let us note the structural finite axiomatizability of a number of classes of groups.

Theorem 4. The following classes of semigroups are structurally finitely axiomatizable: a) the class consisting of the infinite cyclic group; b) the class of torsion-free groups; c) the class of nonperiodic groups; d) the class of nonperiodic Abelian groups; e) the class of torsion-free Abelian groups; f) the class of orderable groups.

The classes of semigroups appearing in Theorems 5 and 6 have generally not been investigated with respect to any structural characterizability, although their structural definability is known.

Theorem 5. The class of commutative semigroups with the cancellation law and without idempotents is structurally axiomatizable.

Theorem 6. The following classes of semigroups are structurally axiomatizable: 1) the class of free semigroups; 2) the class of free commutative semigroups; 3) the class of free semigroups of idempotents.

In proving the first assertion of Theorem 6, one specifies an axiomatizable class of structures K such that $S(\mathfrak{A}) = K \cap S(\mathfrak{B})$, where $\mathfrak{A}$ is the class of free semigroups; moreover, it can be established that K is not finitely axiomatizable. It does not yet follow from this that $\mathfrak{A}$ is not structurally finitely axiomatizable (although it seems probable that this is so), since one cannot assert a priori that there does not exist such a finitely axiomatizable class of structures K_1 that $S(\mathfrak{A}) = K_1 \cap S(\mathfrak{B})$. The corresponding question remains open. Analogous remarks may also be made for the two other assertions of Theorem 6.

Let us note that for any of the classes of semigroups $\mathfrak{A}$ indicated in Theorems 4–6 and in the paragraph preceding Theorem 4 (except for the class of finite cyclic semigroups that are not improper groups), the class $S(\mathfrak{A})$ is arithmetically nonclosed; this is ensured by Corollaries 1–3 of Theorem 1 and by Theorem 3.

Let us now consider the elementary theory of structures of subsemigroups. Reformulating item c) of Theorem 4, one may say that the predicate "to be a nonperiodic subgroup" is formal in the class $S(\mathfrak{B})$. Hence, by virtue of the fact that a periodic subsemigroup in a group is a subgroup, and the predicate "to be a periodic subsemigroup" is formal in the class $S(\mathfrak{B})$ (see the paragraph preceding Theorem 4), it follows that

Lemma 3. The predicate "to be a subgroup" is formal in the class of structures of subsemigroups of all possible groups.

Using Lemma 3, we obtain that for an arbitrary class of groups $\mathfrak{A}$, from the undecidability of the elementary theory of the class of structures of subgroups of groups from $\mathfrak{A}$ there follows the undecidability of the elementary theory of the class $S(\mathfrak{A})$. The latter, in turn, implies the following assertion:

Lemma 4. Let a class of semigroups $\mathfrak{C}$ contain a structurally finitely axiomatizable class of groups $\mathfrak{A}$. If the elementary theory of the class of structures of subgroups of groups from $\mathfrak{A}$ is undecidable, then the elementary theory of the class $S(\mathfrak{C})$ is undecidable.

For each of the classes of groups indicated in items b)–f) of Theorem 4, the elementary theory of the corresponding class of structures of subgroups is undecidable by virtue of the main result of the work of M. I. Kargapolov (^{10}).

Hence, since the class of Abelian torsion-free groups is contained in each of the remaining mentioned classes of groups, by Lemma 4 and item d) of Theorem 4 it follows that

Theorem 7. *The elementary theory of the class* $S(\mathfrak{C})$ *is undecidable for an arbitrary class of semigroups* $\mathfrak{C}$ *containing all Abelian torsion-free groups. In particular, the elementary theory of the class* $S(\mathfrak{B})$ *is undecidable.*

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