# ON FUNDAMENTAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH A BESSEL DIFFERENTIAL OPERATOR

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## Abstract

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*MATHEMATICS*

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# ON FUNDAMENTAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH A BESSEL DIFFERENTIAL OPERATOR

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In the modern theory of differential equations with constant coefficients, questions of the existence of a fundamental solution occupy a central place. There is an extensive literature devoted to the construction of fundamental solutions for broad classes of differential operators. In the present note we intend to indicate fundamental solutions for certain singular differential operators with constant coefficients.

Let

$$\mathcal{L} = \mathcal{L}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, B_y\right) = \sum_{2j+\nu \le 2m} \sum_{i_1,\dots,i_\nu=1}^{n} a_j^{(i_1,\dots,i_\nu)} \frac{\partial^\nu}{\partial x_{i_1} \cdots \partial x_{i_\nu}} B_y^j \tag{1}$$

be a linear differential operator with constant coefficients of order $2m$.

Here and in what follows $B_y^j$ denotes the iteration of the Bessel operator

$$B_y = \frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y} \quad (k > 0,\ y \ge 0). \tag{2}$$

The operator $\mathcal{L}$ is called $B$-elliptic (3) if, for every real vector $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ $(\alpha_{n+1} \ge 0)$, the inequality

$$\left|\mathcal{L}_0\left(i\alpha_1, \dots, i\alpha_n, (i_{n+1})^2\right)\right| \ge \delta|\alpha|^{2m}, \tag{3}$$

holds, where $\delta$ is a positive number and $\mathcal{L}_0$ is the principal part of the operator $\mathcal{L}$.

The subsequent exposition is devoted to the search for fundamental solutions of $B$-elliptic equations.

1. A complex-valued function $\varphi(x,y)$ $(x=(x_1,\dots,x_n))$ is called a **basic function** if it is infinitely differentiable, even in $y$, and satisfies inequalities of the form

$$\left|D^q_{x,y}\varphi\right| \le C_{q\nu}/(1+r^2)^\nu, \tag{4}$$

where

$$r^2=\sum_{i=1}^n x_i^2+y^2$$

for any integers $q\ge 0,\ \nu\ge 0$. The set of all basic functions will be called the **basic space** $S_B$.

For basic functions the mixed Fourier-Bessel transform is defined:

$$F[\varphi]=\int_{R_n}\int_0^\infty \varphi(x,y)e^{ix\sigma}j_\nu(y\tau)y^k\,dx\,dy, \tag{5}$$

$$F^{-1}[\psi]=C_\nu\int_{R_n}\int_0^\infty \psi(\sigma,\tau)e^{-ix\sigma}j_\nu(y\tau)\tau^k\,d\sigma\,d\tau, \tag{6}$$

where $\nu=(k-1)/2>-{}^1\!/_2$, $C_\nu=1/(2\pi)^n2^{2\nu}\Gamma^2(\nu+1)$. In recent times this transform has found wide application in the theory of embeddings for weighted classes ($^4$). The mixed Fourier-Bessel transform maps the space $S_B$ one-to-one onto itself.

By the **convolution** of the basic functions $f$ and $\varphi$ we shall mean the expression

$$f*\varphi=\int_{R_n}\int_0^\infty T^{s,t}_{x,y}f(x,y)\varphi(s,t)t^k\,dt\,ds, \tag{7}$$

where the generalized translation operator $T^{s,t}_{x,y}$ is defined by the formula

$$T^{s,t}_{x,y}f(x,y)= \tag{8}$$

$$=\frac{\Gamma(\nu+1)}{\Gamma({}^1\!/_2)\Gamma(\nu+{}^1\!/_2)}\int_0^\pi f(x_1-s_1,\dots,x_n-s_n,\sqrt{y^2+t^2-2yt\cos\alpha})\sin^{k-1}\alpha\,d\alpha.$$

The operators $T^{s,t}_{x,y}$ and $P(D_x,B_y)$ commute. Here $P(D_x,B_y)$ is an arbitrary polynomial with constant coefficients.

A linear continuous functional on the space $S_B$ will be called a **generalized function**. The space of all functionals will be denoted by $S'_B$. The operation of applying the Bessel operator with respect to the argument $y$ is defined by the formula

$$(B_y f, \varphi) = (f, B_y \varphi). \tag{9}$$

The translation operation for generalized functions is defined by the formula

$$(T^{s,t}_{x,y} f, \varphi) = (f, T^{s,t}_{x,y} \varphi). \tag{10}$$

The convolution of a functional $f$ and a basic function $\varphi$ is defined by the formula

$$f * \varphi = (f(s,t), T^{s,t}_{x,y} \varphi). \tag{11}$$

As the definition of the mixed Fourier–Bessel transform of generalized functions, the equality

$$(\hat{f}, \varphi) = C_\nu (F[f], F[\varphi]) \tag{12}$$

is adopted. Note that $F^{-1}[\delta] = C_\nu$, $F[\delta] = 1$.

Consider the functional $r^\lambda$, acting according to the formula

$$(r^\lambda, \varphi) = \int_{R_n} \int_0^\infty r^\lambda \varphi(x,y) y^k \, dy \, dx. \tag{13}$$

For $\operatorname{Re}\lambda < -(n+k+1)$, the functional (13) is defined by the method of analytic continuation with respect to the parameter $\lambda$. Thus, the generalized function $r^\lambda$ is analytically continued to the whole $\lambda$-plane, except for the points $\lambda = -\gamma, -(\gamma+2), ...$, where $\gamma = n+k+1$, at which it has simple poles.

The generalized function

$$\frac{2r^\lambda}{a_0 \Gamma((\lambda+\gamma)/2)} \tag{14}$$

is an entire analytic function of $\lambda$. The value of this function at the singular points of the numerator and denominator can be found as the ratio of the corresponding residues. Thus, we have

$$\left.\frac{2r^\lambda}{a_0 \Gamma((\lambda+\gamma)/2)}\right|_{\lambda=-\gamma-2p} = (-1)^p \frac{\Delta_B^p \delta(x,y)}{2^p \gamma(\gamma+2) \dots (\gamma+2p-2)}, \tag{15}$$

where

$$a_0 = \int_\Omega \cos^k \varphi_1 \, d\Omega$$

($\Omega$ is the hemisphere $y \geqslant 0$ in $R_{n+1}$),

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + B_y. \tag{16}$$

In particular, for $p = 0$ we have

$$\left. \frac{2r^\lambda}{a_0 \Gamma((\lambda+\gamma)/2)} \right|_{\lambda=-\gamma} = \delta(x, y). \tag{17}$$

Using Sonine’ s second definite integral ($^1$) and Weber’ s improper integral ($^1$), we find that

$$F[r^\lambda] = C_\lambda \rho^{-\lambda-\gamma}, \tag{18}$$

where

$$C_\lambda = 2^{\lambda+\gamma-2} \pi^{n/2-1} \Gamma((k+1)/2) \Gamma((\lambda+\gamma)/2) / \Gamma(-\lambda/2)$$

and $\rho$ is the distance in the corresponding Euclidean space.

Formula (18) makes it possible, with the aid of the mixed Fourier-Bessel transform, to find a solution of the equation $\Delta_B^m u = \delta$, and thereby to construct the fundamental solution for this case. The required fundamental solution has the form

$$u(x, y) = \begin{cases} C_1 r^{2m-\gamma} \ln r, & \text{if } 2m \geqslant \gamma \text{ and } \gamma \text{ is even}, \\ C_2 r^{2m-\gamma}, & \text{in all other cases}. \end{cases} \tag{19}$$

To obtain the fundamental solution with a singularity at an arbitrary point $(s, t)$, one must apply the shift operator $T_{x,y}^{s,t}$ to the function $u(x, y)$. In the case $m = 1$ the fundamental solution was found by A. Weinstein ($^5$).

2. We now turn to the consideration of the general case. The formula for the expansion of $r^\lambda$ into weighted plane waves has the form

$$\frac{1}{\pi^{n/2}\Gamma(k/2)\Gamma((\lambda+1)/2)}\int_{\Omega}|\omega\cdot x|_B^{\lambda}|\omega_{n+1}|^k\,d\Omega=\frac{2r^{\lambda}}{\Gamma((\lambda+\gamma)/2)},\tag{20}$$

where it is put that

$$|\omega\cdot x|_B^{\lambda}=\int_0^{\pi}\left|\sum_{i=1}^{n}\omega_i x_i+\omega_{n+1}y\cos\alpha\right|^{\lambda}\sin^{k-1}\alpha\,d\alpha.$$

Here $\omega=(\omega_1,\dots,\omega_{n+1})$ is an arbitrary vector on the unit sphere $\Omega$ ($|\omega|=1$). In particular, if $\gamma=n+k+1$ and $\lambda=-\gamma$, with $\gamma$ not an even number, then

$$\delta(x,y)=C_1(n,k)\int_{\Omega}|\omega\cdot x|_B^{\lambda}|\omega_{n+1}|^k\,d\Omega\tag{21}$$

with the corresponding value of $C_1(n,k)$. If, however, $\gamma=n+k+1$ is an even number, then

$$|\omega\cdot x|_B^{-\gamma}=\lim_{\lambda\to-\gamma}\frac{|\omega\cdot x|_B^{\lambda}}{\Gamma((\lambda+1)/2)},$$

and then we have

$$\delta(x,y)=C_2(n,k)\int_{\Omega}|\omega\cdot x|_B^{-\gamma}|\omega_{n+1}|^k\,d\Omega.\tag{22}$$

For solving the equation

$$\mathcal{L}(D_x,B_y)u=a_0\delta(x,y),\tag{23}$$

where $\mathcal{L}$ is a linear differential operator of order $2m$ with constant coefficients of $B$-elliptic type (3), we apply the following scheme (see, for example, ($^2$)). We replace the right-hand side of equation (23) by the func-

$$\frac{2r^{\lambda}}{\Gamma((\lambda+\gamma)/2)},$$

and expand the latter into plane waves according to formula (20). Then the problem reduces to the solution of the corresponding ordinary equation. As a result we obtain the following formulas for the fundamental solution ($\mathcal{L}=\mathcal{L}_0$). If $2m\geq n+k+1$ and $n+k+1$ is not an even number, then the fundamental solution has the form

$$u(x,y) = C(n,k)\left\{\int_\Omega \left[\int_0^\pi \left|\sum_{i=1}^n \omega_i x_i + \omega_{n+1} y\cos\alpha\right|^{2m-\gamma} \sin^{k-1}\alpha\, d\alpha\right] \frac{|\omega_{n+1}|^k\, d\Omega}{\mathcal{L}(\omega_1,\dots,\omega_n,\omega_{n+1}^2)}\right\} \tag{24}$$

In the case of even $n+k+1$ we have

$$u(x,y) = C(n,k)\int_\Omega \left[\int_0^\pi \left|\sum_{i=1}^n \omega_i x_i + \omega_{n+1} y\cos\alpha\right|^{2m-\gamma} \times \right.$$

$$\left. \times \ln\left|\sum_{i=1}^n \omega_i x_i + \omega_{n+1} y\cos\alpha\right| \sin^{k-1}\alpha\, d\alpha\right] \frac{|\omega_{n+1}|^k\, d\Omega}{\mathcal{L}(\omega_1,\dots,\omega_n,\omega_{n+1}^2)} \tag{25}$$

with the corresponding value of the constant $C(n,k)$, which we do not write out explicitly. In both cases the fundamental solution is an ordinary function, continuous at the origin.

If $2m < n+k+1$, then the fundamental solution has the form

$$u(x,y) = C(n,k)\int_\Omega |\omega\cdot x|_B^{2m-\gamma} \frac{|\omega_{n+1}|^k\, d\Omega}{\mathcal{L}(\omega_1,\dots,\omega_n,\omega_{n+1}^2)}, \tag{26}$$

and in this case too the fundamental solution is an ordinary function.

To obtain a fundamental solution with singularity at an arbitrary point $(s,t)$, one must apply the shift operator to formulas (24)–(26).

In a neighborhood of the origin the fundamental solution admits the estimates

$$u(x,y) = \begin{cases} O(r^{2m-\gamma}\ln r), & \text{if } 2m \ge \gamma \text{ and } \gamma \text{ is even,} \\ O(r^{2m-\gamma}), & \text{in the remaining cases.} \end{cases} \tag{27}$$

After application of the shift operator $T_{x,y}^{s,t}$, the character of the singularity is smoothed out, and the solution inside the domain $(y>0)$ behaves in the same way as the fundamental solution of an ordinary elliptic equation.

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## REFERENCES

1. Watson, *A Treatise on the Theory of Bessel Functions*, IL, 1949.

2. I. M. Gel' fand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1958.

3. I. A. Kipriyanov, DAN, 158, No. 2 (1964).

4. I. A. Kipriyanov, *Fourier–Bessel Transforms and Embedding Theorems for Weighted Classes*, author's abstract of doctoral dissertation, Moscow, 1964.

5. A. Weinstein, Trans. Am. Math. Soc., 63, 2 (1948).

*Note: Figure translations are in progress. See original paper for figures.*

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