ON THE QUESTION OF SECOND SOUND IN A NONIDEAL BOSE GAS

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Yu. A. TSERKOVNIKOV

ON THE QUESTION OF SECOND SOUND IN A NONIDEAL BOSE GAS

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In work ([1]) an investigation was carried out of the spectrum of elementary excitations of a weakly nonideal Bose gas in the random-phase approximation. It was shown that, in addition to the branch corresponding to ordinary sound, there is another branch with strong damping. It is the analogue of the branch corresponding to second sound in the Green function calculated in the hydrodynamic approximation ([2]). Starting from the same chain of equations for the Green function
$\langle\langle a_q; a_q^+\rangle\rangle = -i\theta(t-t')\langle[a_q(t); a_q^+(t')]\rangle$, as in work ([1]),

$$i\frac{d}{dt}\langle\langle a_q; a_q^+\rangle\rangle = \delta(t-t') + \left(\frac{q^2}{2m} - \mu + \frac{N}{V}\nu(0)\right)\langle\langle a_q; a_q^+\rangle\rangle +$$

$$+\frac{\sqrt{N_0}}{V}\nu(q)\langle\langle \rho_q; a_q^+\rangle\rangle + \frac{1}{V}\sum_{k\neq 0,q}\nu(k)\langle\langle \rho_k a_{q-k}; a_q^+\rangle\rangle,$$

$$i\frac{d}{dt}\langle\langle \rho_q; a_q^+\rangle\rangle = \delta(t-t')\sqrt{N_0} + \langle\langle \rho_q'; a_q^+\rangle\rangle, \tag{1}$$

$$i\frac{d}{dt}\langle\langle \rho_q'; a_q^+\rangle\rangle = \delta(t-t')\frac{q^2}{2m}\sqrt{N_0} + E_q^2\langle\langle \rho_q; a_q^+\rangle\rangle +$$

$$+\langle\langle \rho_q''; a_q^+\rangle\rangle + \frac{1}{V}\sum_{k\neq 0,q}\nu(k)\frac{qk}{m}\langle\langle \rho_k\rho_{-k+q}; a_q^+\rangle\rangle,$$

where N is the total number of particles; N_0 is the number of particles in the condensate;
$\mu = \frac{N}{V}\nu(0) + \frac{1}{V}\sum\nu(k)\langle\rho_{-k}a_k\rangle$ is the chemical potential; $\nu(q)$ is the Fourier component of the interaction potential;

$$\rho_q = \sum a_p^+ a_{p+q}, \qquad \rho_q' = \sum \left(\frac{q^2}{2m} + \frac{pq}{m}\right) a_p^+ a_{p+q},$$

$$\rho_q'' = \sum \frac{pq}{m}\frac{(p+q)q}{m} a_p^+ a_{p+q}, \qquad E_q^2 = \left(\frac{q^2}{2m}\right)^2 + \frac{q^2}{m}\frac{N}{V}\nu(q),$$

let us calculate the corrections to the random-phase approximation arising from the interaction between particles.

From the very beginning we shall neglect the quantity $\langle\langle \rho_q''; a_q^+ \rangle\rangle$, which, generally speaking, is unjustified; however, the results obtained below may be of methodological interest. We shall regard the last terms in the first and third of equations (1) as small. Neglecting these terms, we obtain expressions for the Green functions found in work ([3]). They correspond to the following approximate expressions for the operators in the Heisenberg representation:

$$a_q(t) \simeq \sqrt{1 - \frac{N_0}{N}}\, \alpha_q e^{-i\frac{q^2}{2m}t} + \sqrt{\frac{N_0}{N}} \left(u_q \beta_q e^{-iE_{qt}} + v_q \beta_{-q}^+ e^{iE_{qt}}\right),$$

$$\rho_q(t) \simeq \sqrt{N}(u_q + v_q)\left(\beta_q e^{-iE_{qt}} + \beta_{-q}^+ e^{iE_{qt}}\right), \qquad \rho_q' = i\, d\rho_q/dt, \tag{2}$$

where α_q and β_q are the Bose operators of two subsystems; u_q and v_q are the usual parameters of N. N. Bogolyubov' s canonical transformation ([4]), in which N_0 has been replaced by N.

In equations (1) it is convenient to pass from the operators a_q, ρ_q and ρ_q' to the operators α_q, β_q, using (for $t = 0$) formulas (2) as exact ones. Introduce a vector A_q with components $\alpha_q, \alpha_{-q}^+, \beta_q, \beta_{-q}^+$. For the operator A_q we shall have the equation

$$i\frac{dA_q}{dt} = L_{qA}q + \frac{1}{V}\sum_{k\neq 0,q} v(k) v_{qk}, \tag{3}$$

where L_q is a diagonal matrix with elements $q^2/2m$, $-q^2/2m$, E_q, $-E_q$; v_{qk} is a column with components $\xi_{q,k}, \xi_{-q,-k}^+, \eta_{qk}, \eta_{-q,-k}^+$;

$$\xi_{q,k} = \left(1 - \frac{N_0}{N}\right)^{-1/2} \left(\rho_k a_{q-k} - \frac{\langle \rho_{-k} a_k\rangle}{\sqrt{N_0}} a_q\right) + \left(\frac{N_0}{N}\right)^{1/2} \left(1 - \frac{N_0}{N}\right)^{-1/2} \frac{qk}{q^2}\rho_k \rho_{-k+q},$$

$$\eta_{q,k} = N^{-1/2}(u_q + v_q)\frac{qk}{q^2}\rho_k\rho_{-k+q}. \tag{4}$$

Instead of system (3), for the matrix Green function $G_q = \langle\langle A_q; A_q^+ \rangle\rangle$ we shall have the equations

$$i\frac{dG_q}{dt} = \delta(t-t')I + L_{qG}q + \frac{1}{V}\sum_{k\neq 0,q} v(k)\langle\langle v_{qk}; A_q^+\rangle\rangle,$$

$$i\frac{d}{dt}\langle\langle v_{qk}; A_q^+\rangle\rangle = -i\frac{d}{dt'}\langle\langle v_{qk}; A_q^+\rangle\rangle = \delta(t-t')\langle[v_{qk}; A_q^+]\rangle +$$

$$+\langle\langle v_{qk}; A_q^+\rangle\rangle L_q + \frac{1}{V}\sum_{k'\neq 0,q} v(k')\langle\langle v_{qk}; v_{qk'}^+\rangle\rangle, \tag{5}$$

where I is a diagonal matrix with elements $1, -1, 1, -1$.

Passing to the Fourier components of the Green functions,

$$\langle\langle A(t); B(t')\rangle\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\langle A \mid B\rangle\rangle_E e^{-iE(t-t')}\, dE$$

and eliminating the function $\langle\langle v_{qk} \mid A_q^+\rangle\rangle_E$ from the first equation (5), we obtain

$$\{E - L_q - K_q(E)[I + (E-L_q)^{-1}K_q(E)]^{-1}\}G_q(E) = I, \tag{6}$$

where

$$K_q(E) = \frac{1}{V}\sum_{k\neq 0,q} v(k)\langle[v_{qk}; A_q^+]\rangle + \frac{1}{V^2}\sum_{k,k'\neq 0,q} v(k)v(k')\langle\langle v_{qk} \mid v_{qk'}^+\rangle\rangle_E. \tag{7}$$

If in (6) one neglects the quantity $K_q(E)$, then in the zeroth approximation we shall have $G_q^{(0)}(E) = (E - L_q)^{-1}$, which corresponds to the result obtained in (3). Considering $K_q(E)$ to be a small quantity, in the first approximation we obtain

$$G_q(E) \simeq \{E - L_q - K_q^{(0)}(E)I\}^{-1}I. \tag{8}$$

The components of the matrix $K_q^{(0)}(E)$ are most simply calculated as follows. In the first term of the right-hand side of (7) it is necessary to calculate all commutators. In the second term, instead of the operators $a_q(t)$ and $\rho_q(t)$ entering into ξ_q and η_q (4), one should substitute their approximate expressions (2) and carry out all necessary operations, regarding the operators α_q and β_q as exactly Bose operators. The correlation functions thereby obtained should be replaced by the corresponding correlation functions of the zeroth approximation.

The matrix $G_q^{-1}(E)$ in the approximation under consideration has the form

$$G_q^{-1}(E) = \begin{pmatrix} E - q^2/2m - A_+ & -B & F_+ & F_+ \\ -B & -E - q^2/2m - A_- & F_- & F_- \\ F_+ & F_- & E - E_q - D & -D \\ F_+ & F_- & -D & -E - E_q - D \end{pmatrix}, \tag{9}$$

where $A_\pm \equiv A_q(\pm E)$, $B \equiv B_q(E) = B_q(-E)$, etc.; $A_q(E) = \left(1 - \frac{N_0}{N}\right)^{-1} \times$

$$\times \left[S_q(E) - \frac{N_0}{N}\left(2Q_q(E) - R_q(E)\right)\right]; \quad B_q(E) = \left(1 - \frac{N_0}{N}\right)^{-1}\left[C_q(E) - \frac{N_0}{N}\times\right.$$

$$\left.\times\left(Q_q(E) + Q_q(-E) - R_q(E)\right)\right]; \quad F_q(E) = \left(\frac{N_0}{N}\right)^{1/2}\left(1 - \frac{N_0}{N}\right)^{-1/2}(u_q + v_q)\times$$

$$\times\left[R_q(E) - Q_q(E)\right]; \quad D_q(E) = (u_q + v_q)^2 R_q(E).$$

The quantities $S_q(E)$, $C_q(E)$, $Q_q(E)$, and $R_q(E)$ are expressed in terms of the functions of the zeroth approximation. Setting the explicitly entering temperature θ in them equal to zero, i.e., assuming that the main dependence on θ is determined by the temperature dependence of the number of particles in the condensate N_0, we obtain

$$S_q(E) = \frac{1 - N_0/N}{2V}\sum_{k\neq 0,q}\nu(q-k)\left(1 - \frac{k^2}{2mE_k}\right) + \frac{N_0}{N}\frac{1}{V}\sum_{k\neq 0,q}\nu(q-k)\times$$

$$\times\frac{N}{V}\frac{\nu(k)}{2E_k} + \frac{1}{2V}\sum_{k\neq 0,q}\left(\nu(q-k) - \nu(k)\right)\frac{k^2}{2mE_k} +$$

$$+\frac{N - N_0}{V^2}\sum_{k\neq 0,q}\nu^2(k)\frac{k^2}{2mE_k}\,\frac{1}{E - (q-k)^2/2m - E_k} +$$

$$+\frac{N_0}{2V^2}\sum_{k\neq 0,q}\frac{E\left(K_{qk}^2 - L_{qk}^2\right) + (E_k + E_{q-k})\left(K_{qk}^2 - L_{qk}^2\right)}{E^2 - (E_k + E_{q-k})^2};$$

$$C_q(E) = -\frac{N_0}{N}\frac{1}{V}\sum_{k\neq 0,q}\nu(q-k)\frac{N}{V}\frac{\nu(k)}{2E_k} + \frac{N_0}{V^2}\sum_{k\neq 0,q}\frac{(E_k + E_{q-k})K_{qk}L_{qk}}{E^2 - (E_k + E_{q-k})^2}; \tag{10}$$

$$Q_q(E) = \frac{1}{2V}\sum_{k\neq 0,q}\nu(k)\frac{k^2}{2mE_k} + \frac{1}{2V}\sum_{k\neq 0,q}(\nu(q-k)-\nu(k))\left(1-\frac{qk}{q^2}\right)\frac{k^2}{2mE_k}+$$

$$+\frac{N}{2V^2}\sum_{k\neq 0,q}M_{qk}\frac{E\left(K_{qk}-L_{qk}\right)+\left(E_k+E_{q-k}\right)\left(K_{qk}+L_{qk}\right)}{E^2-\left(E_k+E_{q-k}\right)^2};$$

$$R_q(E) = \frac{1}{V}\sum_{k\neq 0,q}\nu(k)\frac{k^2}{2mE_k} + \frac{1}{V}\sum_{k\neq 0,q}(\nu(q-k)-\nu(k))\left(1-\frac{qk}{q^2}\right)^2\times$$

$$\times\frac{k^2}{2mE_k} + \frac{N}{V^2}\sum_{k\neq 0,q}M_{qk}^2\frac{E_k+E_{q-k}}{E^2-\left(E_k+E_{q-k}\right)^2},$$

where

$$K_{qk} = \nu(k)(u_k+v_k)u_{q-k} + \nu(q-k)(u_{q-k}+v_{q-k})u_k;$$

$$L_{qk} = \nu(k)(u_k+v_k)v_{q-k} + \nu(q-k)(u_{q-k}+v_{q-k})v_k;$$

$$M_{qk} = \left(\frac{qk}{q^2}\nu(k) + \frac{q(q-k)}{q^2}\nu(q-k)\right)(u_k+v_k)(u_{q-k}+v_{q-k}).$$

Inverting the matrix (9), it is easy to find expressions for the Green' s functions

$$G_q^{11}(E) = \langle\langle \alpha_q \mid \alpha_q^+\rangle\rangle, \quad G_q^{21}(E) = \langle\langle \alpha_{-q}^+ \mid \alpha_q^+\rangle\rangle,$$

$$G_q^{31} = \langle\langle \beta_q \mid \alpha_q^+\rangle\rangle, \dots, \quad G_q^{33} = \langle\langle \beta_q \mid \beta_q^+\rangle\rangle, \dots$$

The determinant of the matrix (9) is equal to

$$\Delta_q(E) = \left\{(E-q^2/2m-A_+)(E+q^2/2m+A_-)+B^2\right\}\left\{E^2-E_q^2-2E_{qD}\right\}$$

$$-2E_q\left\{F_+^2(E+q^2/2m+A_-)+F_-^2(-E+q^2/2m+A_+)-2F_+F_-B\right\}, \quad (11)$$

and the equation $\Delta_q(E)=0$ determines the spectrum of the elementary excitations of the Bose system under consideration.

Let us now simplify the expressions obtained. First, in all expressions let us replace the quantities (10) by their values at $q = 0$ and $E = 0$. In doing so we lose the damping and the correction to the effective mass of the particle. Further, in the present work a model of a weakly nonideal Bose gas is considered.

This means that the interaction must be regarded as small, and the density as large[(5)]. Thus, if $v(0) = v(q = 0)$, d is the range of action of the forces and $a = (V/N)^{1/3}$, the following conditions must be satisfied: $v(0)m/d \ll 1$ and the density must be so large, $(a/d) \ll 1$, that the parameter $(d/v(0)m)^{1/2}(a/d)^{3/2}$ may be regarded as small. Using these considerations and replacing the Fourier component of the interaction potential $v(q)$ by a step of height $v(0)$ and width equal to $q_0 = 1/d$, for the quantities $S_0 \equiv S_{p=0}(E = 0)$, C_0, etc., we obtain the expressions:

$$S_0 = C_0 = -\frac{N_0}{2V^2} \sum_{k \neq 0} \frac{v^2(k)}{E_k^3} \left(\frac{k^2}{2m} \right)^2 \simeq -\frac{N_0}{V} v(0)\varepsilon,$$

$$Q_0 \simeq -\frac{N}{V} v(0) \frac{5}{3}\varepsilon, \qquad R_0 \simeq \frac{N}{V} v(0) \frac{13}{15}\varepsilon, \tag{12}$$

where

$$\varepsilon = \frac{1}{64\pi^2} \left(\frac{d}{v(0)m} \right)^{1/2} \left(\frac{a}{d} \right)^{3/2}.$$

Substituting these values into (11), we obtain the equation

$$\Delta_q(E) = \left\{ E^2 - \left(\frac{q^2}{2m} \right)^2 - \frac{q^2}{m} \gamma \frac{N}{V} v(0) \frac{48}{15}\varepsilon \right\} \left\{ E^2 - E_q^2 - \frac{q^2}{m} \frac{N}{V} v(0) \frac{13}{15}\varepsilon \right\} -$$

$$-\gamma \left(\frac{q^2}{m} \right)^2 \left(\frac{N}{V} v(0) \frac{38}{15}\varepsilon \right)^2 = 0, \tag{13}$$

where

$$\gamma = \frac{N_0}{N} \left(1 - \frac{N_0}{N} \right)^{-1}.$$

Equation (13) has as its solutions two phonon branches, $E \sim c_1 q$ and $E \sim c_2 q$. Using the smallness of the parameter ε, for c_1 and c_2 we obtain the expressions

$$c_1^2 = \frac{N}{V} \frac{v(0)}{m} \left(1 + \frac{13}{15}\varepsilon \right), \qquad c_2^2 = \frac{N_0}{N} \left(1 - \frac{N_0}{N} \right)^{-1} \frac{N}{V} \frac{v(0)}{m} \frac{48}{15}\varepsilon. \tag{14}$$

Using the expressions for the components of the matrix $G_q(E)$ found from (9), and the relation between the operators a_q, ρ_q and α_q, β_q (2), we easily find the one-particle, collective, and other Green' s functions. Discarding in $\Delta_q(E)$

(13) the last term as a small term of second order and analogous terms in the numerators of the expressions for G_q^{ij}, we obtain, for example,

$$\langle\langle a_q \mid a_q^+ \rangle\rangle = \left\langle\left\langle \sqrt{1 - \frac{N_0}{N}}\, \alpha_q + \sqrt{\frac{N_0}{N}}\, (u_q \beta_q + v_q \beta_{-q}^+) \right| \sqrt{1 - \frac{N_0}{N}}\, \alpha_q^+ + \sqrt{\frac{N_0}{N}}\, (u_q \beta_q^+ + v_q \beta_{-q}) \left.\right\rangle\right\rangle \simeq \left(1 - \frac{N_0}{N}\right) \frac{E + q^2/2m + mc_2^2}{E^2 - (q^2/2m)^2 - c_2^2 q^2} + \frac{N_0}{N} \frac{E + q^2/2m + mc_1^2}{E^2 - (q^2/2m)^2 - c_1^2 q^2}. \tag{15}$$

The velocity of ordinary sound c_1, as is seen from (14), is practically independent of temperature. The velocity c_2 goes to zero when $N_0 = 0$.

Let us note that, owing to the simplifications made at the beginning, the ratio of the velocities of first and second sound has turned out to be incorrect. The correct ratio can be obtained only in the hydrodynamic approximation[(2)] or on the basis of the kinetic equation.

In conclusion, I express my deep gratitude to N. N. Bogolyubov for discussion of the work.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.