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ON THE IDENTITY PROBLEM IN FINITELY DEFINED SEMIGROUPS

(Presented by Academician P. S. Novikov on 27 I 1966)

I. In connection with the theorem of A. A. Markov–E. L. Post ([2,6]) on the existence of a finitely defined semigroup with an unsolvable identity problem, the question arose of constructing such semigroups that are as simple as possible. The strongest results in this direction were obtained by G. S. Tseitin ([5]) and D. Scott ([7]), who constructed semigroups with an unsolvable identity problem that are given by only seven defining relations.

In the present note the semigroup Π_6^4 is considered, given in the alphabet $\{a, c, e, f\}$ by the six defining relations $ac = ca,\ aff = ffa,\ fac = cfa,\ eca = ce,\ efffa = ffe,\ aacc = aacce.$

Theorem 1. *In the semigroup* Π_6^4 *the identity problem for a certain fixed word is unsolvable.*

Replacing in the semigroup of G. S. Tseitin ([5]) the relation $cca = ccae$ by two relations $aaac = aacce$ and $bacc = bacce$, we obtain the semigroup Π_8^5. Then, relying on the theorem of P. S. Novikov ([3]) on the existence of a finitely defined group with an unsolvable identity problem, we shall prove that in the semigroup Π_8^5 the identity problem for a certain fixed word is unsolvable. Then we consider a homomorphism φ of the semigroup Π_8^5 into the semigroup Π_6^4, given on generators by $\varphi(a) = a,\ \varphi(b) = fa,\ \varphi(c) = c,\ \varphi(d) = ff,\ \varphi(e) = e$. Finally, we prove the following: if X and Y are words in the semigroup Π_8^5, then $X = Y$ in the semigroup Π_8^5 if and only if $\varphi(X) = \varphi(Y)$ in the semigroup Π_6^4.

From Theorem 3 of the work ([4]) of P. S. Novikov and S. I. Adian it follows:

Theorem 2. *There exists an algorithm which, for every semigroup* S *and arbitrary words* X *and* Y *in this semigroup, constructs words* $\widehat{X}$ *and* $\widehat{Y}$ *in the alphabet* $\{a, c, e, f\}$ *such that* $X = Y$ *in the semigroup* S *if and only if* $\widehat{X} = \widehat{Y}$ *in the semigroup* Π_6^4.

II. A semigroup defined by a finite alphabet

$$a_1, a_2, \dots, a_n \tag{1}$$

and by a finite system of defining relations

$$\{A_i = 1 \quad (i = 1, 2, \dots, k), \tag{2}$$

such that

$$[A_i^0] \geq l \quad (i = 1, 2, \dots, k), \tag{3}$$

will be called a (k, l)-semigroup.

A word X in a semigroup Π is **left-invertible** (**right-invertible**) if there exists a word Y in the semigroup Π such that $YX = 1$ $(XY = 1)$ in the semigroup Π. A word X is **two-sided invertible** in the semigroup Π if X is invertible both on the right and on the left. If in a (k, l)-semigroup Γ every word is two-sided invertible, then Γ will be called a (k, l)-group.

S. I. Adian, in Chapter III of the work (^{1}), investigated the connection between the solvability of the identity problem in (k, l)-groups and the solvability of the identity and divisibility problems in homogeneous (k, l)-semigroups, i.e. in (k, l)-semigroups,

in which the left-hand sides of the defining relations have the same length. In the present note Theorems 3, 4, 5 are proved, analogous to the theorems of S. I. Adian, but already for arbitrary (k, l)-semigroups. In addition, the connection between the solvability of systems of equations in (k, l)-groups and (k, l)-semigroups is investigated.

Theorem 3. *If the natural numbers k and l are such that there exists an algorithm solving the identity problem in any (k, l)-group, then one can specify an algorithm solving the identity problem in every (k, l)-semigroup.*

We shall say that an ordered pair of words A and B **intersect** if $A = MN$, $B = NL$, where N and ML are nonempty words.

We shall say that the nonempty words $X_1, X_2, \dots, X_m$ are obtained from the words $Y_1, Y_2, \dots, Y_p$ by **partitioning** if: a) each Y_i is graphically equal to the product $X_{\Delta(i,1)}, X_{\Delta(i,2)}, \dots, X_{\Delta(i,s_i)}$, where $1 \leq \Delta(i,1), \Delta(i,2), \dots, \Delta(i,s_i) \leq m$; b) for every X_j $(j = 1, 2, \dots, m)$ there exists a Y_i such that X_j is graphically equal to one of $X_{\Delta(i,1)}, X_{\Delta(i,2)}, \dots, X_{\Delta(i,s_i)}$; c) $X_i \neq X_j$ for $i \neq j$.

Consider a (k, l)-semigroup Π given by the alphabet (1), the relations (2), and the condition (3).

A finite set of words $B_1, B_2, \dots, B_r$ in the alphabet (1) is called a **system of B-words** of the semigroup Π if: 1) $B_1, B_2, \dots, B_r$ are obtained from $A_1, A_2, \dots, A_k$

by partitioning (here $A_1, A_2, \dots, A_k$ are the left-hand sides of the defining relations (2)); 2) $B_1, B_2, \dots, B_r$ do not intersect with one another.

Lemma 1. *There exists an algorithm which, for every (k, l)-semigroup, constructs some system of B-words of this semigroup.*

The operation of extraction by a semigroup Π and a system of B-words of a semigroup Γ: to each B-word of the semigroup Π, B_i $(i = 1, 2, \dots, r)$, put in correspondence a letter β_i $(i = 1, 2, \dots, r)$; the semigroup Γ is given by the alphabet $\beta_1, \beta_2, \dots, \beta_r$ and the defining relations

$$\{\beta_{\Delta(i,1)}\beta_{\Delta(i,2)} \cdots \beta_{\Delta(i,s_i)} = 1 \quad (i = 1, 2, \dots, k).$$

A finite set of words $C_1, C_2, \dots, C_s$ in the alphabet (1) is called a **system of C-words** of the semigroup Π if $C_1, C_2, \dots, C_s$ is a system of B-words of Π and 3) the semigroup Γ, extracted by the semigroup Π and the system of B-words $C_1, C_2, \dots, C_s$, is a (k, l)-group.

Lemma 2. *There exists an algorithm which, for every (k, l)-semigroup, constructs some system of C-words of this semigroup.*

By the hypothesis of Theorem 3, in the group Γ, extracted by the (k, l)-semigroup Π and the system of C-words, the identity problem is solvable.

Let X be a word in the alphabet (1) and $X = h_1 C_{i_1} C_{i_2} \cdots C_{i_v} h_2$, where h_1, h_2 are nonempty. Then $\rho(X) = h_1 C_{j_1} C_{j_2} \cdots C_{j_w} h_2$, where $[X^\partial \supseteq [\rho(X))^\partial$ and $\beta_{i_1}\beta_{i_2} \cdots \beta_{i_v} = \beta_{j_1}\beta_{j_2} \cdots \beta_{j_w}$ in the group Γ.

Lemma 3. *There exist only finitely many Y such that $\rho(X) = Y$.*

Let C_i be some C-word of the semigroup Π. Each $\rho(\rho(\rho \cdots \rho(C_i) \cdots))$ will be called a **c_i-word** of the semigroup Π.

Lemma 4. *If the hypotheses of Theorem 4 are satisfied, then there exists an algorithm which, for every (k, l)-semigroup and for every C-word C_i of this semigroup, constructs all c_i-words $c_i^{(1)}, c_i^{(2)}, \dots, c_i^{(r_i)}$ of this semigroup.*

A (k, l)-semigroup V with already constructed C-words is called **selected** if:

a) $[C_i^\partial = [c_i]^\partial$ for $i = 1, 2, \dots, s;\ j = 1, 2, \dots, r_i$;

b) no $c_u^{(t)}$ belongs to the set of words $c_p^{(1)}, c_p^{(2)}, \dots, c_p^{(r_p)}$ for $p \neq u$;

c) no $c_{u_1}^{(t_1)}$ and $c_{u_2}^{(t_2)}$ intersect with one another.

Lemma 5. If the conditions of Theorem 3 are satisfied, then there exists an algorithm which, for every (k, l)-semigroup Π, constructs a chosen (k, l)-subsemigroup V isomorphic to Π.

Lemma 6. If the conditions of Theorem 4 are satisfied, then there exists an algorithm solving the identity problem in every chosen (k, l)-semigroup.

Theorem 4. If the natural numbers k and l are such that there exists an algorithm solving the identity problem in any (k, l)-group, then one can specify an algorithm solving the problem of left (right) divisibility in every (k, l)-semigroup.

The **maximal subgroup** of a given semigroup is the subgroup generated by all two-sided invertible elements of this semigroup.

Theorem 5. If the natural numbers k and l are such that there exists an algorithm solving the identity problem in any (k, l)-group, then one can specify an algorithm which, for every (k, l)-semigroup Π, constructs a (k, l)-group isomorphic to the maximal subgroup of Π.

A **system of equations** in a (k, l)-semigroup Π, given in the alphabet $a_1, a_2, \dots, a_n$, is a system of expressions

$$W_i = T_i \qquad (i = 1, 2, \dots, m), \tag{4}$$

where W_i are words in the alphabet $a_1, a_2, \dots, a_n, X_1, X_2, \dots, X_p$, and T_i are words in the alphabet $a_1, a_2, \dots, a_n$. A solution of this system is taken to be words $V_1, V_2, \dots, V_p$ in the alphabet $a_1, a_2, \dots, a_n$, whose substitution respectively for $X_1, X_2, \dots, X_n$ in the expressions (4) transforms each W_i into a word equal to T_i in the semigroup Π.

Theorem 6. If the natural numbers k and l are such that there exists an algorithm checking the solvability of any system of equations in any (k, l)-group, then one can specify an algorithm checking the solvability of any system in any (k, l)-semigroup.

In conclusion I express my sincere gratitude to S. I. Adyan for his constant assistance in the work.

Note added in proof. When the paper had already been submitted for publication, the author obtained the following result. In the semigroup Π_5^3, given in the alphabet $\{e, f, g\}$ by the defining relations $ggff = ffgg$, $fgggff = gfffgg$, $eggff = ffe$, $efgggff = gffe$, $ffggffffgg = ffggffffgge$, the identity problem is undecidable. Moreover, the author became aware that Yu. Matiyasevich obtained a similar result.

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Note: Figure translations are in progress. See original paper for figures.