ENTIRE FUNCTIONS OF FINITE ORDER WITH AN INFINITE SET OF DEFICIENT VALUES

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ENTIRE FUNCTIONS OF FINITE ORDER WITH AN INFINITE SET OF DEFICIENT VALUES

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A well-known conjecture of R. Nevanlinna states ([1,2]) that an entire function of finite order ρ has only a finite number (not more than $[2\rho]+1$) of deficient values. For the case $\rho \le 1/2$ this conjecture has been confirmed. Edrei and Fuchs showed, in particular ([3]), that an entire function of order $\rho \le 1/2$ cannot have finite deficient values. It turns out that in the case $\rho > 1/2$ R. Nevanlinna' s conjecture is false. Moreover, the following is true.

Theorem. For an arbitrary sequence of complex numbers $\{a_k\}_1^\infty$ and for any $\rho > 1/2$, there exists an entire function of order ρ and of normal type for which the numbers a_k, $k = 1, 2, ...$, are deficient.

1°. The proof of this theorem is based on the following two lemmas from approximation theory.

Lemma 1. Let L denote a rectifiable curve joining the points a and b. Then for arbitrary numbers $\varepsilon > 0$ and $d > 0$ there exists a polynomial $P(z)$ such that

$$\left|\frac{1}{a-z} - P\left(\frac{1}{z-b}\right)\right| < \varepsilon \tag{1}$$

for all points z lying outside the d-neighborhood of the curve L, and

$$\left|P\left(\frac{1}{z-b}\right)\right| < \exp\left[(1+|\ln \varepsilon d|)\exp\frac{5\text{ length } L}{d}\right] \tag{2}$$

outside the circle $|z-b| < d$.

Lemma 2. Let $1/2 < \rho \le 1$, $0 < \alpha < \dfrac{\pi}{\rho}(\rho - 1/2)$, and let the function $F(z)$ be analytic in the angle $|\arg z| \le \beta/2$ $(\beta > \alpha)$ and satisfy $|F(z)| < \exp[\text{const}(|z|+1)^\rho]$. Then for any $\varepsilon > 0$ there exists an entire function $G(z)$ of order ρ and of normal type satisfying the inequality $|F(z) - G(z)| < \varepsilon \exp(-|z|^\rho)$, $|\arg z| \le \alpha/2$.

We note that Lemma 1 is a special case of an important lemma of M. V. Keldysh ([4]), while Lemma 2 is an immediate consequence of a theorem of M. V. Keldysh ([5]) on approximation in an angle by entire functions of restricted growth.

2°. In proving the theorem it is sufficient to restrict ourselves to the case when $1/2 < \rho \le 1$. Indeed, if $\rho > 1$, choose a natural number N such that $\rho \le N < 2\rho$. Having constructed an entire function $G(z)$ of order ρ/N with deficient values $\{a_k\}_1^\infty$, we may then consider the entire function $G(z^N)$ of order ρ with the same defects.

3°. Now take an arbitrary number α, $0 < \alpha < \dfrac{\pi}{\rho}(\rho - 1/2)$, and a sequence $\{\theta_k\}_{-\infty}^{+\infty}$ such that $\theta_{-k} = -\theta_k$, $\theta_k \uparrow +\infty$ as $k \uparrow +\infty$. Denote by $\Delta_k(\varepsilon)$ the angle $|\arg z - \theta_k| \le \varepsilon\alpha_k$, where $\alpha_k = \min(\theta_{k+1} - \theta_k, \theta_k - \theta_{k-1})$, by $\sigma_n(\delta)$ $(\delta \ge 0)$ the annulus $(1-\delta)2^n \le |z| \le (1+\delta)2^{n+1}$, and put $E_{k,n}(\varepsilon,\delta) = \Delta_k(\varepsilon) \cap \sigma_n(\delta)$. Let now

now $\{n_k\}_1^\infty$ is some sequence of even natural numbers, $n_k \uparrow +\infty$ as $k \uparrow +\infty$. Setting $n_{-k} = n_k + 1$, consider the closed sets

$$E(\varepsilon,\delta) = \bigcup_{|k|=1}^{+\infty} \bigcup_{m=0}^{+\infty} E_{k,n_k+2m}(\varepsilon,\delta),$$

$$E_0 = E(1/4, 0), \qquad E_1 = E(1/2, 1/8), \qquad E_2 = E(3/4, 1/4),$$

so that $E_0 \subset E_1 \subset E_2$. Setting further

$$\varepsilon_k = \exp(-2^{12} a_k^{-1}), \qquad k = \pm 1, \pm 2, \dots,$$

we shall assume that the n_k increase so rapidly that

$$\prod_{|k|=1}^{+\infty} \prod_{n=n_k}^{+\infty} \frac{1+\delta_{k,n}}{1-\delta_{k,n}} < {}^{16}/_{15}, \qquad \text{where } \delta_{k,n} = \exp(-\varepsilon_k 2^{n\rho}), \tag{3}$$

$$\max(a_k^{-1}, 2^{n_k}, |a_k|) < \exp(\varepsilon_k 2^{n_k\rho - 1}), \qquad k = 1, 2, \dots, \tag{4}$$

$$\int_{\partial E_1} |\zeta^{-2} d\zeta| < 1. \tag{5}$$

4°. We now denote by $\gamma_{k,n}^i$ $(i = 1, 2;\ k = \pm 1, \pm 2, \dots;\ n = n_k + 2m;\ m = 1, 2, \dots)$ the boundary of the set $E_i \cap E_{k,n}(3/4, 1/4)$. The curve

$$l_{k,n}^i: \ z = (1 + 2^{i-4}) 2^{n+1} e^{i\theta}, \quad (-1)^{n_k}\theta_n \le (-1)^{n_k}\theta \le \pi,$$

joins the curve $\gamma^i_{k,n}$ with the point $-(1+2^{i-4})2^{n+1}$. By $D^i_{k,n}$ we denote the d-neighborhood of the curve $\gamma^i_{k,n} \cup l^i_{k,n}$, where $d = a_k 2^{n-4}$. It is easy to verify that $E_{i-1} \subset CD^i_{k,n}$, and that the length $\gamma^i_{k,n} \cup l^i_{k,n} < 2^{n+4}$. Take an arbitrary point $\zeta \in \gamma^i_{k,n}$ and apply Lemma 1 to the domain $D^i_{k,n}$, putting

$$a = \zeta, \qquad b = -(1+2^{i-4})2^{n+1}, \qquad \varepsilon = 4^{-n-2}\exp(-5\varepsilon_k 2^{n\rho}).$$

We may suppose that the function

$$Q_i(\zeta, z) \equiv P(1/(z-b))$$

is analytic in z in the half-plane $\operatorname{Re} z > -1$, is defined and piecewise constant in ζ (uniformly with respect to z) on the set ∂E_i, and satisfies, by virtue of (1), (2), and (4), the inequalities

$$|Q_i(\zeta, z) - 1/(\zeta - z)| < 4^{-n-2}\exp(-5\varepsilon_k 2^{n\rho}) \quad \text{for } \zeta \in \gamma^i_{k,n},\ z \in CD^i_{n,k}, \tag{6}$$

$$|Q_i(\zeta, z)| < \exp(\varepsilon_k^{1/2} 2^{n\rho+4}) \quad \text{for } \zeta \in \gamma^i_{k,n},\ \operatorname{Re} z > -1. \tag{7}$$

5°. We now define two functions $\varphi(z)$ and $\psi(z)$, analytic on the set E_2, by putting

$$\varphi(z) = a_{|k|}, \quad \psi(z) = \exp(-\varepsilon_k z^\rho), \qquad \text{when } z \in \Delta_k(3/4),\ k = \pm 1, \pm 2, \dots. \tag{8}$$

Lemma 3. There exists a function $\omega(z)$, analytic in the half-plane $\operatorname{Re} z > -1$, such that

$$1 < |\omega(z)/\psi(z)| < 2 \quad \text{for } z \in E_1, \tag{9}$$

$$|\omega(z)| < \exp[(|z|+1)^\rho] \quad \text{for } \operatorname{Re} z > -1. \tag{10}$$

Proof. Consider the function

$$\omega_{k,n}(z) = 1 + \frac{1}{2\pi i}\int_{\gamma^2_{k,n}} [\psi(\zeta) - 1] Q_2(\zeta, z)\, d\zeta, \tag{11}$$

analytic in the half-plane $\operatorname{Re} z > -1$. Observing now that, by the Cauchy formula,

$$\psi_{k,n}(z) = 1 + \frac{1}{2\pi i}\int_{\gamma^2_{k,n}} \frac{\psi(\zeta)-1}{\zeta - z}\,d\zeta = \begin{cases}\psi(z), & \text{for } z \in E_{k,n}(1/2, 1/8),\\ 1, & \text{for } z \in CE_{k,n}(3/4, 1/4),\end{cases}$$

and taking into account that the length of $\gamma^2_{k,n} < 2^{n+3}$, from (6) and (11) we obtain

$$|\omega_{k,n}(z) - \psi_{k,n}(z)| < {}^1/_4 \exp(-5\varepsilon_k 2^{n\rho}) \quad \text{for } z \in CD^i_{k,n},$$

whence

$$ {}^3/_4 < |\omega_{k,n}(z)/\psi(z)| < {}^5/_4 \quad \text{for } z \in E_{k,n}(1/2, 1/8), \tag{12}$$

$$|\omega_{k,n}(z) - 1| < \exp(-\varepsilon_k 2^{n\rho}) \quad \text{for } z \in CD^2_{k,n} \cap CE_{k,n}(3/4, 1/4). \tag{13}$$

Inequality (13) holds, in particular, for $z \in E_1 \setminus E_{k,n}(1/2, 1/8)$.

The growth of the function $\omega_{k,n}(z)$ is bounded, by virtue of (7), (11), and (4), by the inequality

$$|\omega_{k,n}(z)| < \exp\left(\varepsilon_k^{1/2} \cdot 2^{n\rho+5}\right), \qquad \operatorname{Re} z > -1. \tag{14}$$

Now define the desired function $\omega(z)$ by the formula

$$\omega(z) = {}^3/_2 \prod_{|k|=1}^{+\infty} \prod_{m=0}^{+\infty} \omega_{k,n_k+2m}(z).$$

The convergence of this product follows from (13) and (3). To obtain estimate (9), it suffices to use (12), (13), and (3). Finally, to estimate the growth of $\omega(z)$, take an arbitrary number z, $\operatorname{Re} z > -1$. Let $z \in \sigma_N(0)$. Then $z \in CD^2_{k,n} \cap CE_{k,n}(3/4, 1/4)$ for all n, except, possibly, for the cases $n = N-1, N, N+1$. Taking (13), (14), and (3) into account, we obtain:

$$|\omega(z)| < 2 \prod_{|k|=1}^{+\infty} \prod_{n=N-1}^{N+1} |\omega_{k,n}(z)| \exp(2^{N\rho}) < \exp[(|z|+1)^\rho].$$

Lemma 4. *There exists a function $F(z)$, analytic in the angle $|\arg z| \le \pi/2$, such that*

$$|\varphi(z) - F(z)| < {}^1/_2 |\psi(z)| \qquad \text{for } z \in E_0,$$

$$|F(z)| < \exp[\mathrm{const}\cdot(|z|+1)^\rho] \qquad \text{for } |\arg z| \le \pi/2.$$

Proof. We note that from (6) the estimate follows

$$|\theta_1(\xi,z) - 1/(\xi - z)| < |\psi(\xi)|/|\xi|^2, \qquad \xi \in \partial E_1, \tag{15}$$

valid for $z \in E_0$, and also in the disk $|z| \le {}^1/_4|\xi|$. The growth of the function $Q_1(\xi,z)$ is bounded, by virtue of estimate (7), by the inequality

$$|Q_1(\xi,z)| < \exp(|\xi|^\rho) \qquad \text{for } \xi \in \partial E_1,\ \operatorname{Re} z > -1. \tag{16}$$

Now denote

$$\Gamma_n = \bigcup_{\substack{(k,m)\\ n_k+2m\le n}} \gamma_{k,n_k+2m} \qquad \text{for } n \ge n_1$$

(so that $\partial E_1 = \bigcup_{n=n_1}^{+\infty}\Gamma_n$) and consider the sequence of functions analytic for $\operatorname{Re} z > -1$,

$$H_n(z) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{\varphi(\xi)}{\omega(\xi)} Q_1(\xi,z)\,d\xi, \tag{17}$$

where $\omega(z)$ is the function constructed in Lemma 3. From inequalities (4) and (10) (taking account of (8)) it follows that

$$|\varphi(\xi)/\omega(\xi)| < 1/|\psi(\xi)|^2, \qquad \xi \in E_1. \tag{18}$$

Take a natural number N, and let $|z| \le 2^N,\ m > n > N+1$. Then

$$0 \equiv \frac{1}{2\pi i}\int_{\Gamma_m\setminus\Gamma_n} \frac{\varphi(\xi)}{\omega(\xi)}\frac{d\xi}{\xi - z},$$

whence, taking (15), (17), (18), and (5) into account, we obtain:

$$|H_m(z) - H_n(z)| < \frac{1}{2\pi}\int_{\partial E_1\setminus\Gamma_n} |\xi^{-2}\,d\xi| \to 0 \qquad \text{as } n \to \infty. \tag{19}$$

It follows that the sequence $H_n(z)$ converges uniformly in every disk to some function $H(z)$ analytic for $\operatorname{Re} z > -1$. Similarly, taking into account the formula

$$\frac{\varphi(z)}{\omega(z)} = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{\varphi(\xi)}{\omega(z)}\frac{d\xi}{\xi - z}, \qquad z \in E_0, \quad |z| \le 2^n,$$

for an arbitrary point $z \in E_0$ we have the estimate

$$\left|H(z) - \frac{\varphi(z)}{\omega(z)}\right| = \lim_{n\to\infty}\left|H_n(z) - \frac{\psi(z)}{\omega(z)}\right| \le \frac{1}{2\pi}\int_{\partial E_1} |\xi^{-2}\, d\xi| < {}^1/_4. \tag{20}$$

Let us now estimate the growth of the function $H(z)$. Let $z \in \sigma_{N-1}(0)$. If $\zeta \in \Gamma_{N+2}$, then $|\zeta| \leqslant 18|z|$. Taking now (19), (17), (18), (16), and (15) into account, we obtain

$$|H(z)| < 1 + |H_{N+2}(z)| < 1 + \max_{\zeta\in\Gamma_{N+2}}\left|\frac{\zeta}{\psi(\zeta)}\right|^2 \cdot |Q_1(\zeta, z)| <$$

$$< 1 + \max_{|\zeta|\leqslant 18|z|} |\zeta|^2 \exp(2|\zeta|^\rho) < \exp[\mathrm{const}\cdot(|z|+1)^\rho]. \tag{21}$$

By virtue of (20), (9), and (21), the function $F(z) = H(z)\omega(z)$ satisfies all the conditions of Lemma 4.

6°. From Lemmas 4 and 2 there follows the existence of an entire function $G(z)$ of order ρ and of normal type, satisfying the inequality

$$|G(z) - \varphi(z)| < |\psi(z)|, \qquad z \in E_0. \tag{22}$$

The function $G(z)$ is the required function with an infinite set of deficient values $\{a_k\}_1^\infty$. Indeed, by virtue of the definition of the set E_0 and of the function $\varphi(z)$, for any $k = 1, 2, \ldots$, when $r \geqslant 2^{n_k}$, there exists an arc $\lambda_{k,r}$ of the circumference $|z| = r$, $\lambda_{k,r} \subset E_0$, such that $\varphi(z) = a_k$ for $z \in \lambda_{k,r}$, and the length $\lambda_{k,r} = \dfrac{\alpha_k}{2}r$. Then from (22) we obtain that

$$|G(z) - a_k| < |\psi(z)| < \exp\left(-\frac{\varepsilon_k}{2}r^\rho\right), \qquad z \in \lambda_{k,r},\ r \geqslant 2^{n_k}.$$

Hence, in the generally accepted notation, we have

$$m(r, a_k, G) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{|G(re^{i\theta}) - a_k|}\, d\theta > \frac{\alpha_k\varepsilon_k}{8\pi}r^\rho > \varepsilon_k^2 r^\rho \quad \text{for } r \geqslant 2^{n_k},$$

and since $T(r, G) \leqslant \ln^+ M(r, G) < \sigma r^\rho$ for $r \geqslant 1$, we finally obtain

$$\delta(a_k, G) = \lim_{r\to\infty} [m(r, a_k, G)/T(r, G)] \geqslant \varepsilon_k^2/\sigma > 0.$$

The theorem is completely proved.

Remark. Analysis of the example constructed above gives some grounds to suppose that, for an arbitrary entire function $G(z)$ of finite order, the condition

$$\sum_{0<\delta(a,G)<1} \frac{1}{\ln \delta^{-1}(a, G)} < +\infty$$

is satisfied.

This condition cannot be further strengthened, since in our example $1/\ln \delta^{-1}(a_k, G) > \text{const} \cdot \alpha_k > 0$, where of the numbers a_k it is required only that the series $\sum_{k=1}^{\infty} \alpha_k$ converge.

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Note: Figure translations are in progress. See original paper for figures.