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Abstract

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ON HIGHER APPROXIMATIONS OF THE AVERAGING METHOD OF N. N. BOGOLYUBOV –N. M. KRYLOV

(Presented by Academician N. N. Bogolyubov, 31 I 1966)

The role of the asymptotic methods of N. N. Bogolyubov–N. M. Krylov (see ([1−3])) in the study of solutions of differential equations with a small parameter is well known. However, the justification of these methods is often carried out under superfluous restrictions. To get rid of these restrictions, as it seems to the authors, is of considerable interest. For the first asymptotic approximation the most general theorem was obtained in the work of M. A. Krasnosel' skii and S. G. Krein ([4]). In the present work a generalization of this theorem to the case of higher asymptotic approximations is proposed.

1. Let E be a finite-dimensional space, and let D be a bounded domain in E. Consider in E the ordinary differential equations

$$dx/dt = X_0(t,x) + \varepsilon X_1(t,x) + ... + \varepsilon^k X_k(t,x) + \varepsilon^k \omega(t,x,\varepsilon); \tag{1}$$

$$d\bar{x}/dt = X_0(t,x) + \varepsilon X_1(t,x) + ... + \varepsilon^k X_k(t,x). \tag{2}$$

Here $0 \le t \le T$, $x \in D$, ε is a scalar parameter; the right-hand sides of equations (1)–(2) are continuous in x and measurable in t and ε. We shall be interested in the question of the order of closeness, as $\varepsilon \to 0$, of the solutions $x(t)$ and $\bar{x}(t)$ of equations (1) and (2), satisfying the initial condition $x(0) = \bar{x}(0) = x_0$ (the sets of these solutions will be denoted by $\mathfrak{M}(\varepsilon, T)$ and $\mathfrak{M}_k(\varepsilon, T)$, respectively).

We shall say that $Z(t,x) \in H_\theta$ ($\theta \in [0,1]$), if

$$\|Z(t,x) - Z(t,y)\| \le q(t)\|x-y\|^\theta,$$

$$\|Z(t,x+z) - Z(t,x) - Z(t,y+z) + Z(t,y)\| \le q(t)\varphi(\|x-y\|)\|z\|^\theta$$

and if there exists an operator $W(t,x,h)$ $(0 \le t \le T,\ x \in D,\ h \in E)$ such that

$$\|W(t,x,h)\| \le q(t)\|h\|^{\theta},$$

$$\|W(t,x,h) - W(t,y,h)\| \le q(t)\varphi(\|x-y\|)\|h\|^{\theta},$$

$$\lim_{\varepsilon\to 0}\int_0^t \frac{Z(\tau,x+\varepsilon h) - Z(\tau,x)}{\varepsilon^{\theta}}\,d\tau = \int_0^t W(\tau,x,h)\,d\tau.$$

Here $q(t)$ is a function summable on $[0,T]$, and $\varphi(u) \to 0$ as $u \to 0$. The operator $W(t,x,h)$ is, obviously, homogeneous of order θ in h. It is easy to see that $W(t,x,h) \equiv 0$ if $Z(t,x) \in H_{\theta}$ and $\theta < 1$.

Theorem 1. Let $X_i(t,x) \in H_{(k-i)/k}$ $(i = 0,\dots,k)$. Suppose the operator $\omega(t,x,\varepsilon)$ $(0 \le t \le T,\ x \in D,\ 0 \le \varepsilon \le \varepsilon_0)$ satisfies the conditions

$$\|\omega(t,x,\varepsilon)\| \le q(t),$$

$$\|\omega(t,x_1,\varepsilon) - \omega(t,x_2,\varepsilon)\| \le q(t)\varphi(\|x_1 - x_2\|).$$

where $q(t)$ is a summable function on $[0,T]$, $\varphi(u) \to 0$ as $u \to 0$, and

$$\lim_{\varepsilon\to 0}\left\|\int_0^t \omega(\tau,x,\varepsilon)\,d\tau\right\| = 0.$$

Finally, suppose that the Cauchy problem $dx/dt = X_0(t,x)$, $x(0) = x_0$, has in the domain D a unique solution defined on $[0,T]$. Then

$$\lim_{\varepsilon\to 0}\ \sup_{x(t)\in\mathfrak{M}(\varepsilon,T),\ \bar{x}(t)\in\mathfrak{M}_k(\varepsilon,T)}\ \max_{0\le t\le T}\frac{\|x(t) - \bar{x}(t)\|}{\varepsilon^k} = 0.$$

2. Consider the ordinary differential equation

$$dx/dt = \varepsilon X_0(x) + \dots + \varepsilon^k X_{k-1}(x) + \varepsilon^{k+1}X_k(t,x,\varepsilon); \tag{3}$$

here $0 \le t < \infty$, $x \in D$, ε is a scalar parameter; the right-hand side of equation (3) is continuous in x and measurable in t and ε. Suppose that

$$\lim_{\varepsilon\to 0}\left\|\frac{\varepsilon}{t}\int_0^{t/\varepsilon} X_k(s,x,\varepsilon)\,ds - \bar{X}_k(x)\right\| = 0$$

(in this case we shall say that the operator $X_k(t,x,\varepsilon)$ converges, in the averaged sense, to the operator $\bar{X}_k$), and, together with equation (3), consider the equation

$$dx/dt = \varepsilon X_0(x) + \dots + \varepsilon^k X_{k-1}(x) + \varepsilon^{k+1}\bar{X}_k(x). \tag{4}$$

We shall be interested in the question of the order of closeness on the interval $[0, T/\varepsilon]$ between solutions of equations (3) and (4) satisfying the same initial condition $x(0) = x_0$ (the sets of these solutions, as above, will be denoted by $\mathfrak{M}(\varepsilon, T/\varepsilon)$ and $\mathfrak{M}_k(\varepsilon, T/\varepsilon)$).

Theorem 2. Let $X_i(x) \in H_{(k-i)/k}$ $(i = 0, \dots, k-1)$. Let the operator $X_k(t,x,\varepsilon)$ satisfy the conditions

$$\|X_k(t,x,\varepsilon)\| \le q(t,\varepsilon), \tag{5}$$

$$\|X_k(t,x,\varepsilon) - X_k(t,x,\varepsilon)\| \le q(t,\varepsilon)\varphi(\|x_1 - x_2\|), \tag{6}$$

where $q(t,\varepsilon)$ is some function satisfying the condition $q(t/\varepsilon,\varepsilon) \le q_0(t)$, and $q_0(t)$ is a function summable on every finite interval; $\varphi(u) \to 0$ as $u \to 0$. Suppose that the operator $X_k(t,x,\varepsilon)$ converges, in the averaged sense, to the operator $\bar{X}_k(x) \in H_0$. Finally, suppose that the Cauchy problem $dx/dt = X_0(x)$, $x(0) = x_0$, has in the domain D a unique solution defined on the half-axis $0 \le t < \infty$.

Then for any $T > 0$ the equality

$$\lim_{\varepsilon\to 0} \sup_{x(t)\in\mathfrak{M}(\varepsilon,T/\varepsilon),\ \bar{x}(t)\in\mathfrak{M}_k(\varepsilon,T/\varepsilon)} \max_{0\le t\le T/\varepsilon} \frac{\|x(t) - \bar{x}(t)\|}{\varepsilon^k} = 0$$

holds.

3. Consider the differential equation

$$dx/dt = \varepsilon X_0(t,x) + \varepsilon^2 X_1(t,x) + \dots + \varepsilon^{kX}_{k-1}(t,x) + \varepsilon^{k+1}X_k(t,x,\varepsilon). \tag{7}$$

By means of the change of variable [1–3]

$$x = y + \varepsilon U_1(t,y) + \dots + \varepsilon^k U_k(t,y)$$

we pass from equation (7) to an autonomous equation, up to terms of order $k+1$ in ε,

$$dy/dt = \varepsilon Y_0(y) + \varepsilon^2 Y_1(y) + \dots + \varepsilon^{kY}_{k-1}(y) + \varepsilon^{k+1}Y_k(t,y,\varepsilon). \tag{8}$$

Here the operators $Y_i(y)$, $U_{i+1}(t,y)$ $(i=0,\dots,k-1)$ are determined successively are determined from the equalities

$$Y_i(y)=\lim_{T\to\infty}\frac{1}{T}\int_0^T F_i(\tau,y)\,d\tau; \tag{9}$$

$$U_{i+1}(t,y)=\int_0^t \{F_i(\tau,y)-Y_i(y)\}\,d\tau, \tag{10}$$

where

$$F_\alpha(t,y)=\sum_{i+j+l=\alpha} Q_{lP_{ij}}-\sum_{0\le m\le\alpha-1} Q_{\alpha-m}\frac{\partial U_{m+1}}{\partial t}\quad(\alpha=0,\dots,k-1);$$

the operators Q_i are determined from the expansion

$$\left[I+\varepsilon\frac{\partial U_1}{\partial y}+\dots+\varepsilon^k\frac{\partial U_k}{\partial y}\right]^{-1}=\sum_{i=0}^{\infty} Q_i\varepsilon^i$$

(it is assumed that, for small ε, the norm of the operator $\varepsilon\,\partial U_1/\partial y+\dots+\varepsilon^k\,\partial U_k/\partial y$ is small); the operators P_{ij} are determined from the expansions

$$X_i(t,y+\varepsilon U_1+\dots+\varepsilon_k^{kU})=\sum_{j=0}^{k-i} P_{ij}\varepsilon^j+\widetilde{P}_i(\varepsilon)\varepsilon^{k-i+1};$$

the operator $Y_k(t,y,\varepsilon)$ is determined by the formula

$$Y_k(t,y,\varepsilon)=F_k(t,y)+L(t,y,\varepsilon),$$

where

$$F_k(t,y)=\sum_{i+j+l=k,\ i\ne k} Q_{lP_{ij}}+X_k(t,y)-\sum_{0\le m\le k-1} Q_{k-m}\frac{\partial U_{m+1}}{\partial t};$$

$$\begin{aligned}L(t,y,\varepsilon)&=X_k(t,y+\varepsilon U_1+\dots+\varepsilon_k^{kU})-X_k(t,y)\\&+\left[\sum_{1\le l<\infty} Q_l\varepsilon^l\right]X_k(t,y+\varepsilon U_1+\dots+\varepsilon_k^{kU})+\sum_{\substack{0\le i\le k-1\\0\le l<\infty}} Q_l\widetilde{P}_i(\varepsilon)\varepsilon^l\\&+\sum_{\substack{0\le i+j\le k-1\\i+j+l>k}} Q_{lP_{ij}}\varepsilon^{i+j+l-k}-\sum_{\substack{0\le m\le k-1\\m+l>k}} Q_l\frac{\partial U_{m+1}}{\partial t}\varepsilon^{m+l-k}.\end{aligned}$$

To investigate equation (8), Theorem 2 is applied. In doing so one is able to verify only stronger properties of the operators $Y_i(y)$ $(i = 0, \dots, k-1)$ than the conditions H_θ. Namely, it can be shown that $Y_i(y)$ are continuously differentiable $k-i$ times if the operators $X_i(t,x)$ are continuously differentiable $k-i$ times and if, as $t \to \infty$, the operators

$$\frac{1}{t}\int_0^t F_i(\tau, y)\, d\tau$$

converge to the operator $Y_i(y)$ together with derivatives up to order $k-i$, uniformly on every closed subset D (in this case, for small ε, the norm of the operators $\varepsilon\, \partial U_1/\partial y + \dots + \varepsilon^k\, \partial U_k/\partial y$ is small, and therefore the change of variables described above is justified). If, moreover, the operator $Y_k(t, y, \varepsilon)$ satisfies inequalities (5), (6) and tends, in the mean, to some operator $\overline{Y}_k(y) \in H_0$, then we shall say that for equation (7) the conditions $(\mathcal{P}_k)$ are fulfilled.

In particular, the conditions $(\mathcal{P}_k)$ are fulfilled if all the operators X_i $(i = 0, \dots, k)$, together with derivatives up to order $k-i$, are bounded; if the operators U_i, successively determined by formulas (10), are bounded; and if, finally, there exists an averaging of the operator F_k. Verification of these boundedness conditions is especially simple when the operators X_i are periodic or almost periodic in t.

Suppose that the conditions $(\mathcal{P}_k)$ are satisfied. Let $y(t)$ be a solution of equation (8), and let $\bar{y}(t)$ be a solution of the equation

$$dy/dt = \varepsilon Y_0(y) + \dots + \varepsilon^{kY}_{k-1}(y) + \varepsilon^{k+1}\bar{Y}_k(y),$$

satisfying the same initial condition $y(0) = \bar{y}(0) = x_0$. Put

$$x(t) = y(t) + \varepsilon U_1[t, y(t)] + \dots + \varepsilon^k U_k[t, y(t)]; \tag{11}$$

$$\bar{x}(t) = \bar{y}(t) + \varepsilon U_1[t, \bar{y}(t)] + \dots + \varepsilon^k U_k[t, \bar{y}(t)]. \tag{12}$$

The function $\bar{x}(t)$ is called an **asymptotic approximation** (in the sense of Bogolyubov–Krylov) of order $k+1$ to the function $x(t)$, the exact solution of equation (7), satisfying the initial condition $x(0) = x_0$. We shall denote by $\mathfrak{M}(\varepsilon, T/\varepsilon)$ the set of solutions of equation (7) defined on $[0, T/\varepsilon]$ and satisfying the initial condition $x(0) = x_0$, and by $\mathfrak{M}_k(\varepsilon, T/\varepsilon)$ the set of the corresponding asymptotic approximations of order $k+1$.

We shall call the domain D **regular** if there exists a constant l such that any two points $x, y \in D$ can be joined by a simple curve of length not exceeding $l|x-y|$.

Theorem 3. *Let, for equation (5), the conditions $(\mathcal{P}_k)$ be satisfied, and let the domain D be regular.*

Then for every $T > 0$ the equality

$$\lim_{\varepsilon\to 0} \sup_{x(t)\in\mathfrak{M}(\varepsilon,T/\varepsilon),\ \bar{x}(t)\in\mathfrak{M}_k(\varepsilon,T/\varepsilon)} \max_{0\le t\le T/\varepsilon} \frac{\|x(t)-\bar{x}(t)\|}{\varepsilon^k} = 0$$

holds.

4. The theorems stated in the preceding sections carry over to equations in infinite-dimensional spaces under the additional assumption that the right-hand sides of these equations are the sum of an operator satisfying the Lipschitz condition and a compact operator.

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Note: Figure translations are in progress. See original paper for figures.