ON THE STABILITY OF CLASSES OF WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR AN EQUATION WELL-POSED IN THE SENSE OF I. G. PETROVSKII

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.944

MATHEMATICS

V. M. BOROK, Ya. I. ZHITOMIRSKII

ON THE STABILITY OF CLASSES OF WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR AN EQUATION WELL-POSED IN THE SENSE OF I. G. PETROVSKII

(Presented by Academician I. G. Petrovskii, 23 VI 1965)

The purpose of this note is to study questions of the well-posed solvability of the Cauchy problem in various classes of functions for equations of the form

$$\frac{\partial u(x,t)}{\partial t} = \sum_{j=0}^{p} a_j(x) \frac{\partial^j}{\partial x^j} u(x,t), \qquad -\infty < x < \infty, \quad 0 \le t \le T < \infty, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x). \tag{2}$$

In the case when the coefficients of equation (1) are constant ($a_j(x) = a_j =$ const), and this equation itself is well-posed in the sense of I. G. Petrovskii, various classes of well-posedness of the solution of the Cauchy problem are known (1–3). We assume that the coefficients of equation (1) satisfy the conditions $\lim_{|x|\to\infty} a_j(x) = a_j$ $(j = 0, 1, \dots, p-1)$, $a_p(x) \equiv a_p =$ const, and that the "limiting" equation

$$\frac{\partial u(x,t)}{\partial t} = \sum_{j=0}^{p} a_j \frac{\partial^j}{\partial x^j} u(x,t) \tag{3}$$

is well-posed in the sense of I. G. Petrovskii. In this case it turns out that the classes of well-posed solvability of the Cauchy problem (1)–(2) are the same as for the "limiting" equation (3) with constant coefficients. We establish the well-posed solvability of problem (1)–(2) in the following classes of functions: a) functions growing at infinity no faster than a fixed power of $|x|$; b) functions growing exponentially on one of the half-axes and, on the other, as in case a) (for odd p); c) $L_2(-\infty, \infty)$. Classes a) and b) cannot be essentially enlarged, as is seen from the corresponding result for equation (3) $^{(2)}$. In case c) we give an estimate of the norm (in L_2) of the solution in terms of the norm of the initial function and its derivatives. In addition, we construct a fundamental solution of problem (1)–(2) and give its estimates.

The indicated results are established under certain assumptions on the character of the convergence of $a_j(x)$ to a_j $(j = 0, 1, \dots, p-1)$, assumptions which do not require the well-posedness in the sense of I. G. Petrovskii of equation (1) with "frozen" coefficients.

For what follows it is convenient for us to write equation (1) in the form

$$\frac{\partial u(x,t)}{\partial t} = P_0\left(\frac{\partial}{\partial x}\right) u(x,t) + P_1\left(x, \frac{\partial}{\partial x}\right) u(x,t),$$

where

$$P_0\left(\frac{\partial}{\partial x}\right) = \sum_{j=0}^{p} a_j \frac{\partial^j}{\partial x^j}, \qquad P_1\left(x, \frac{\partial}{\partial x}\right) = \sum_{j=0}^{p-1} c_j(x) \frac{\partial^j}{\partial x^j}, \qquad c_j(x) = a_j(x) - a_j,$$

$$j = 0, 1, \dots, p-1.$$

p. 1. Let $G(x,t)$ be the Green' s function of the Cauchy problem for equation (3). In (4) estimates have been obtained for the function $G(x,t)$ and its derivatives in the strip $0 < t \le T,\ -\infty < x < \infty$:

$$\left|\frac{\partial^r}{\partial x^r} G(x,t)\right| \le g\, \frac{(1+|x|)^{\alpha_r}}{t^{\beta_r}}, \qquad r = 0, 1, \dots, p-2, \tag{4}$$

where the exponents α_r and β_r depend on the form of equation (3), $\alpha_0 = 0,\ \alpha = \max_r \alpha_r < 1,\ \beta_r = \dfrac{r}{p-1} + \dfrac{1}{p} \le 1 - \dfrac{1}{p(p-1)} = \beta$.

Theorem 1. Let the coefficients $c_j(x)$ of the operator $P_1(x, \partial/\partial x)$ satisfy the conditions

$$\int_{-\infty}^{\infty} \left|\frac{d^r}{dx^r} c_j(x)\right| (1+|x|)^{2\alpha}\, dx < \infty,$$

$$j = 0, 1, \dots, p-1; \qquad r = 0, 1, \dots, l \quad (l \ge p),$$

and let the coefficient $c_{p-1}(x)$ be bounded (for $-\infty < x < \infty$) together with derivatives of sufficiently high order. Then there exists a fundamental solution $\Phi(x, \xi, t)$ of the Cauchy problem for equation (1); it has the form

$$\Phi(x,\xi,t) = G(x-\xi,t) + W(x,\xi,t), \tag{5}$$

where $G(x,t)$ is the Green' s function of the Cauchy problem for equation (3), and for $W(x,\xi,t)$ the estimate holds

$$|W(x,\xi,t)| \le C(1+|\xi|)^{\alpha} t^{1-\beta-4/p},$$

$$-\infty < x,\xi < \infty, \quad 0 < t \le T, \quad C = C(T).$$

The function $W(x,\xi,t)$ has the form

$$W(x,\xi,t) = \int_0^t d\tau \int_{-\infty}^{\infty} G(x-y,t-\tau)\varphi(y,\xi,\tau)\,dy;$$

the function $\varphi(x,\xi,t)$ is the solution of the integral equation

$$\varphi(x,\xi,t) = P_1\left(x,\frac{\partial}{\partial x}\right) G(x-\xi,t) +$$

$$+\int_0^t d\tau \int_{-\infty}^{\infty} P_1\left(x,\frac{\partial}{\partial x}\right) G(x-y,t-\tau)\varphi(y,\xi,\tau)\,dy.$$

This equation is solved by the method of successive approximations. All estimates are based on inequalities (4).

Let now p be odd, $P_0(i\sigma) = ia_p\sigma^p + \dots$.

Lemma 1. If $a_p x > 0,\ 0 < t \le T$, then for the function $G(x,t)$ the estimates

$$\left|\frac{\partial^r}{\partial x^r} G(x,t)\right| \le \frac{C_1}{t^{(r+1)/p}} \exp\{-c_2|x|^{p'} t^{-1/(p-1)}\}, \quad c_2 > 0, \tag{6}$$

$$r = 0,1,\dots,p-2; \qquad p' = p/(p-1).$$

The estimates (6) are obtained rather simply from the formula

$$\frac{\partial^r}{\partial x^r} G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\sigma)^r \exp\{tP_0(i\sigma) + i\sigma x\}\,d\sigma$$

by passing in it to integration along a certain straight line parallel to the real axis. As the asymptotics of the function $G(x,t)$ for $t \to +0,\ |x| \to \infty$ show (3), these estimates cannot be substantially improved.

With the help of Lemma 1 one establishes

Theorem 2. Let p be odd and let the coefficients of the operator $P_1(x, \partial/\partial x)$ satisfy the conditions of Theorem 1. Then for the function $W(x, \xi, t)$ in (5)

the estimate is valid

$$|W(x, \xi, t)| \leq C_3(T)(1 + |\xi|)^{\alpha} t^{-\beta-1/p} \exp\{-c_2|x - \xi|^{p'} t^{-1/(p-1)}\},$$

$$0 < t \leq T, \qquad -\infty < \xi \leq x < \infty.$$

p. 2. In what follows the following two basic lemmas are used.

Lemma 2. Suppose: 1) $s \geq 0$; 2) the coefficients of the operator $P_1(x, \partial/\partial x)$ satisfy the conditions

$$\int_{-\infty}^{\infty} (1 + |x|)^{\alpha+\max(a,s)} \left| \frac{d^r}{dx^r} c_i^j(x) \right| dx < \infty,$$

$$j = 0, 1, \dots, p - 1, \qquad r = 0, 1, \dots, l,$$

and the coefficient $c_{p-1}(x)$ is bounded together with a certain number (depending on l) of derivatives; 3) the initial function $u_0(x)$ and its derivatives up to order h (depending on p, s, and l) satisfy the conditions

$$u_0(x) \equiv 0 \quad \text{for } |x - \nu| > 1; \qquad \left| \frac{d^r}{dx^r} u_0(x) \right| \leq U_0, \quad r = 0, 1, \dots, h.$$

Then the solution $u(x, t)$ of the Cauchy problem (1)–(2) has the form

$$u(x, t) = \int \Phi(x, \xi, t) u_0(\xi)\, d\xi \tag{7}$$

and for it the estimates

$$\left| \frac{\partial^r}{\partial x^r} u(x, t) \right| \leq U_0 C_4(T) \left[Q_{\nu,s} + \frac{1}{(1 + |x - \nu|)^{s+2}} \right], \tag{8}$$

$$r = 0, 1, \dots, l,$$

are valid, where

$$\sum_{\nu=-\infty}^{\infty} (1 + |\nu|)^s Q_{\nu,s} < \infty.$$

Lemma 3. If the conditions of Lemma 2 are fulfilled and p is odd, then estimate (8) is sharpened as follows:

$$\text{for } x \geq \nu + 1, \qquad a_p(\nu+1) \leq 0, \qquad r = 0, 1, \dots, l$$

$$\left| \frac{\partial^r}{\partial x^r} u(x,t) \right| \leq U_0 C_5(T) \exp\left\{ -\frac{c_2}{2} |x - \nu - 1|^{p'} t^{-1/(p-1)} \right\}.$$

The proof of these lemmas is based on estimates (4) and (6) and on Theorems 1 and 2.

p. 3. **Theorem 3.** Suppose the conditions 1) and 2) of Lemma 2 are fulfilled, and suppose the initial function $u_0(x)$ and its derivatives up to some order h (h depends on p, l, and s) satisfy the estimate

$$\left| \frac{d^r}{dx^r} u_0(x) \right| \leq U_0 (1 + |x|)^s, \qquad r = 0, 1, \dots, h. \tag{9}$$

Then there exists a unique solution $u(x,t)$ of the Cauchy problem (1)–(2), satisfying the estimate

$$\left| \frac{\partial^r}{\partial x^r} u(x,t) \right| \leq A U_0 (1 + |x|)^s, \qquad r = 0, 1, \dots, l. \tag{10}$$

For the proof of the theorem the initial function $u_0(x)$, with the aid of a partition of unity, is represented as the sum of a series of finite functions:

$$u_0(x) = \sum_{\nu=-\infty}^{\infty} u_{0\nu}(x).$$

Let $u_\nu(x,t)$ be the solution of the Cauchy problem (1)–(2) with initial function $u_{0\nu}(x)$. Then

$$u(x,t) = \sum_{\nu=-\infty}^{\infty} u_\nu(x,t). \tag{11}$$

The convergence of the series (11) and the estimate (10) of its sum are proved with the aid of Lemma 2. The uniqueness of the solution thus obtained in the class of functions satisfying estimate (10) is proved by the method used in (^{4}).

We note that for $s = 0$ Theorem 3 guarantees the correct solvability of problem (1)–(2) in the class of bounded functions. In this case the conditions on the coefficients $c_j(x)$ coincide with the conditions of Theorem 1.

We also note that the power order of growth of the initial function $u_0(x)$ and of the solution $u(x,t)$ of problem (1)–(2), as is seen from (9)–(10), coincide. Thus Theorem 3 refines one of the results of (^{2}), where in the case $c_j(x) \equiv 0$, $j = 0, 1, \ldots, p-1$, under the fulfillment of conditions (9), estimates (10) were established with s replaced by $s+2$.

Theorem 4. *Suppose that conditions 1) and 2) of Lemma 2 are satisfied; p is an odd number and the initial function $u_0(x)$ and its derivatives up to some order $h = h(p,s,l)$ satisfy the conditions*

$$\left|\frac{d^r}{dx^r} u_0(x)\right| \le U_0 \begin{cases} \exp(d|x|^{p'}), & a_p x \le 0, \\ (1+|x|)^s, & a_p x \ge 0, \end{cases}$$

$$d > 0, \qquad r = 0, 1, \ldots, h.$$

Then there exists a unique solution $u(x,t)$ of the Cauchy problem (1)–(2), for which, for some T_1, $0 < T_1 \le T$ (T_1 depends on d), the estimates

$$\left|\frac{\partial^r}{\partial x^r} u(x,t)\right| \le BU_0 \begin{cases} \exp(d_1|x|^{p'}), & a_p x \le 0, \\ (1+|x|)^s, & a_p x \ge 0, \end{cases}$$

$$d_1 > 0, \qquad r = 0, 1, \ldots, l, \qquad 0 \le t \le T_1$$

hold.

The proof of Theorem 4 is analogous to the proof of Theorem 3; it uses Lemma 3.

Theorem 5. *Let the coefficients $C_j(x)$ of the operator $P_1(x, \partial/\partial x)$ satisfy the conditions of Theorem 1 and, in addition,*

$$\int_{-\infty}^{\infty} \left|\frac{d^l}{dx^l} c_j(x)\right|^2 (1+|x|)^{2\alpha}\, dx < \infty, \qquad j = 0, 1, \ldots, p-1.$$

Then, if the initial function $u_0(x)$ and its derivatives up to some order h (depending on p and l) are square-integrable on the whole axis, there exists a unique solution $u(x,t)$ of the Cauchy problem (1)–(2), which, together with its derivatives up to order l, is also square-integrable on the whole axis for every $t > 0$, and the estimate

$$\left\|\frac{\partial^r}{\partial x^r} u(x,t)\right\|_{L_2} \le D \max_{0 \le j \le p+l-2} \left\|\frac{d^j}{dx^j} u_0(x)\right\|_{L_2},$$

$$r = 0, 1, \ldots, l, \qquad 0 \le t \le T < \infty$$

is valid.

The proof of Theorem 5 uses the result of Theorem 3 (for $s = 0$). As in Theorem 3, the solution $u(x,t)$ is written in the form of the sum of the series (11), and then the norm of each term of this series is estimated.

Kharkov State University
named after A. M. Gorky

Received
22 VI 1965

CITED LITERATURE

[1] I. G. Petrovskii, Bull. Moscow Univ., Section A, 1, no. 7 (1938).
[2] I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, vol. 3, Moscow, 1958.
[3] M. V. Fedoryuk, Mat. sbornik, 62 (104), 4 (1963).
[3] Ya. I. Zhitomirskii, Izv. Vyssh. Uchebn. Zaved., Mathematics, 4 (17) (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.