# NONLINEAR TRANSFORMATIONS OF PROBABILITY MEASURES IN FUNCTIONAL SPACES

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## Abstract

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*MATHEMATICS*

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# NONLINEAR TRANSFORMATIONS OF PROBABILITY MEASURES IN FUNCTIONAL SPACES

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1. Numerous applications of the theory of random processes, especially in radio engineering, lead to the problem of determining the characteristics of a random process obtained at the output of a certain device when a given random process is fed to the input of this device. This problem is solved rather simply if the device is linear. For nonlinear transformers, general methods for solving the posed problem have not yet been developed. An exception is the Wiener process, for which nonlinear transformations were studied in the work of R. H. Cameron and W. T. Martin ($^1$), and also in the monograph of N. Wiener ($^2$). Let us also note the book of R. Deutsch ($^3$), in which various particular methods are collected that have been used to solve concrete problems of applied significance.

In general terms the problem can be formulated as follows. Let $\mu$ be a measure in a functional space corresponding to the initial process; $\nu$ a measure corresponding to the transformed process; the problem consists in constructing the measure $\nu$ from the measure $\mu$ and the given transformation (for ways of specifying measures in functional spaces, see ($^4$)). In ($^1$), for the case when $\mu$ is the measure corresponding to the Wiener process, for a broad class of transformations the density $d\nu/d\mu$ of the measure $\nu$ with respect to the measure $\mu$ is found explicitly. Using this result, the computation of the characteristics of the transformed process (they can always be expressed through integrals with respect to the measure $\nu$) can be reduced to the computation of certain other characteristics of the process at the output.

In the present note an extension of this method to processes of a very general form is considered (we shall not touch here at all on the method of the work ($^2$), which consists in expanding the transformation in an orthogonal system of functionals). Let us also note that for Gaussian processes (though not under the most general assumptions concerning the transformations) results of the indicated type were obtained in the work of A. D. Shatashvili ($^5$). The only restriction on the initial process will be that its sample functions be square integrable and, consequently, that the corresponding measure can be considered in some separable Hilbert space.

2. Let $H$ be a separable Hilbert space, $\mathfrak{B}$ the $\sigma$-algebra of subsets of $H$ that is the completion, with respect to a measure $\mu$ given on $\mathfrak{B}$, of the $\sigma$-algebra of all Borel subsets of $H$. Let, further, $T(x)$ $(x \in H)$ be a $\mathfrak{B}$-measurable mapping of $H$ into $H$. Denote by $\nu$ the measure obtained from $\mu$ under the mapping $T$: $\nu(A) = \mu(T^{-1}(A))$, $A \in \mathfrak{B}$, where $T^{-1}(A)$ is the full inverse image of $A$. We shall be interested in conditions under which the measure $\nu$ is absolutely continuous with respect to the measure $\mu$, and in the form of the density $d\nu/d\mu$. It turns out that an essential role in computing $d\nu/d\mu$ for a broad class of transformations is played by the form of this density for the simplest transformations—shifts. Put

$$S_a x = x + a, \qquad \mu_a(A) = \mu(S_a^{-1} A).$$

Suppose that there exists a linear manifold $L$ in $H$ such that $\mu_a$ is absolutely continuous with respect to $\mu$ for all $a \in L$. We shall need the function

$$\rho(a, x) = \frac{d\mu_a}{d\mu}(x),$$

defined for all $a \in L$ and, for each $x$, $\mu$-almost everywhere for all $x \in H$.

In what follows it will be assumed that the mapping $T(x)$ has a strong first variation, i.e., that there exists a family of linear operators $\delta T(x)$ such that

$$\|T(x+u) - T(x) - \delta T(x) u\| = o(\|u\|) \qquad (u \in H).$$

If the operator $C$ has the form $C = E + B$, where $E$ is the identity and $B$ is a linear completely continuous operator, then we shall put

$$|\det C| = \left(\prod_k |1 + \lambda_k|\right)^{1/2},$$

where $\lambda_k$ is the sequence of all eigenvalues of the symmetric completely continuous operator $B + B^* + BB^*$ ($B^*$ is the operator adjoint to $B$), provided the infinite product converges or diverges to zero.

**Theorem.** *Suppose the following conditions are satisfied:*

1) *there exists an orthonormal sequence* $e_1, e_2, \ldots$, *complete in* $L$, *such that*

$$\lim_{n\to\infty} \rho\left(\sum_{k=1}^{n} (a, e_k) e_k,\ x_0 + \sum_{k=1}^{n} (x, e_k) e_k\right) = \rho(a, x_0 + x)$$

*for all $a \in L$ and $\mu$-almost all $x$ and $x_0$ from $H$;*

2) *the transformation $T(x)$ is invertible, i.e., there exists a transformation $U(x)$ such that $T(U(x)) = U(T(x)) = x$; moreover, there exists $\delta U(x)$, and the quantity $|\det \delta U(x)|$ is defined for $\mu$-almost all $x$, is positive, and is a $\mathfrak{B}$-measurable function of $x$;*
3) *for $\mu$-almost all $x \in H$ the expression $\rho(U(x) - x, x)$ is defined and $\mathfrak{B}$-measurable.*

*Then the measure $\nu$ is absolutely continuous with respect to the measure $\mu$, and*

$$\frac{d\nu}{d\mu}(x) = \rho(x - U(x), x)|\det \delta U(x)|. \tag{1}$$

**Proof.** Let us first suppose that $H$ is a finite-dimensional space and that $L$ coincides with $H$. Then the measure $\mu$ will be absolutely continuous with respect to Lebesgue measure (this follows from the fact that the measure $\mu$ is absolutely continuous with respect to all its shifts), i.e. $\mu(dx) = p(x)\,dx$. Making a change of variables in the integral, we obtain

$$\int f(x)\nu(dx) = \int f(T(x))\mu(dx) = \int f(T(x))p(x)\,dx = \int f(y)p(U(y))\left|\frac{dU(y)}{dy}\right| dy,$$

where $|dU(y)/dy|$ is the modulus of the Jacobian of the transformation, which in this case is equal to $|\det \delta U(y)|$. Next note that, in our notation, $\rho(a, x) = p(x - a)/p(x)$. Therefore

$$p(U(y))/p(y) = \rho(y - U(y), y),$$

and hence

$$\int f(x)\nu(dx) = \int f(x)\rho(x - U(x), x)|\det \delta U(x)|p(x)\,dx$$

$$= \int f(x)\rho(x - U(x), x)|\det \delta U(x)|\mu(dx).$$

Thus, in the finite-dimensional case formula (1) is established. The extension of this formula to the infinite-dimensional case is carried out by a limiting passage (for this one may use, for example, Lemma 4 § 2 Ch. 4 from (6)).

**Remark.** The condition of invertibility of $T(x)$ may be replaced by the condition that $T(x)$ have a finite number of preimages. In this case, in formula (1) the expression $\rho(x - U(x), x)|\det \delta U(x)|$ must be replaced by

$$\sum \rho(x - U(x), x)|\det \delta U(x)|,$$

where the summation is over all preimages $U(x)$ of the point $x$ under the mapping $T(x)$; one should also assume the existence and measurability of the indicated sum.

**3.** Let us apply the theorem to the case when $\mu$ is a Gaussian measure. Suppose that

$$\int (x,z)\,d\mu = 0, \qquad \int (x,z)(x,y)\,d\mu = (Bz,y),$$

where $B$ is a symmetric completely continuous operator. Let $e_1, e_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ be, respectively, the sequences of eigenvectors and eigenvalues of the operator $B$ ($e_k$ is an orthonormal sequence). It follows from Chapter 4 (⁷) that in the present case the manifold $L$ coincides with the set of those vectors $a$ for which

$$\sum_k \frac{1}{\lambda_k}(a,e_k)^2 < \infty,$$

$$\rho(a,x)\exp\left\{\sum_{k=1}^{\infty}\frac{(a,e_k)(x,e_k)}{\lambda_k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\lambda_k}(a,e_k)^2\right\}.$$

It is evident that, for the sequence $e_k$, condition 1) of the theorem is satisfied. Therefore, if condition 2) is satisfied for the transformation $T(x)$, and the series

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k}(x-U(x),e_k)(x,e_k)$$

and

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k}(x-U(x),e_k)^2$$

converge $\mu$-almost everywhere, then

$$\frac{d\nu}{d\mu}(x) = |\det \delta U(x)|\exp\left\{\sum_{k=1}^{\infty}\frac{(x-U(x),e_k)(x,e_k)}{\lambda_k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(x-U(x),e_k)^2}{\lambda_k}\right\}. \tag{2}$$

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*Note: Figure translations are in progress. See original paper for figures.*

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