ON THE STABILITY OF THE CHARACTERISTIC EXPONENTS OF LIMIT SOLUTIONS OF LINEAR SYSTEMS

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A. M. Lyapunov (see ([1]), p. 184) introduced the concept of the characteristic exponent of a solution of the system

$$dx/dt = A(t)x \quad \text{in } E^n \quad (\|A(t)\| \leq a;\ t \geq t_0). \tag{1}$$

The totality of the characteristic exponents of system (1), as Perron showed (see ([1]), p. 199), is, generally speaking, unstable (and therefore in fact does not characterize the system): there exist (for certain systems (1)) systems

$$dy/dt = A(t)y + \varphi(y,t); \qquad \|\varphi(y,t)\| \leq g(t)\|y\| \tag{2}$$

with arbitrarily small $\sup_{t \geq t_0} g(t)$, for which the totality of characteristic exponents differs greatly from the totality of characteristic exponents of system (1). After the appearance of this result of Perron, the theory of characteristic exponents began to develop in the following direction: one sought a class of systems (1) for which the totality of characteristic exponents is stable. After the verification of the hypothesis arising from Perron' s example and seeming erroneous (see ([8])), in works of B. F. Bylov, R. E. Vinograd, and D. M. Grobman ([5,6,9,10,12]) these systems were found—these are systems with the condition of "integral separation." The final result may be found in ([9,10]).

In the present work an attempt is made to carry out an investigation in another direction: to "correct" the concept of the totality of characteristic exponents so that it gives a stable characteristic of any system (1). In this, the most complete result is obtained for systems with slowly varying coefficients (see Theorem 4 below), to which also belongs the system in Perron' s example (see ([1]), p. 199).

We make essential use of the concept of a limit solution (for the definition of a limit solution that we use, see ([13])). Let us indicate the physical meaning of limit solutions. It consists in the following. Suppose system (1) describes some material (as they say, "physical") system. In measuring the quantity x we are limited in time, and the accuracy of the measurements is bounded below; therefore we shall not distinguish the trajectory of a limit solution from the trajectory of a true solution if t is greater than some t_1 (the system is sufficiently "old"). Both this physical meaning and the results of ([13]) show that it is inadvisable to consider only true solutions of nonautonomous systems without including limit solutions in the discussion. True and limit solutions will be called generalized solutions.

From Theorem 1 of the work ([14]) it is seen that, for the asymptotics of systems (1) and (2) to coincide, it is sometimes necessary to require exponential decay of $g(t)$. The following theorem shows that the totality of limit solutions of system (1) is a much more stable characteristic of the system.

Theorem 1. Let $g(t) = g_1(t) + g_2(t)$, where

$$g_1(t) \xrightarrow[t\to\infty]{} 0, \qquad \int^{\infty} g_2(\tau)\, d\tau < \infty.$$

Then the set of limiting solutions of system (2) coincides with the set of limiting solutions of system (1).

The proof is readily obtained from the well-known formula

$$y(t) - x(t) = \int_{t_1}^{t} X(t)X^{-1}(\tau)\varphi(y(\tau), \tau)\, d\tau.$$

Let $e_1(t), e_2(t), \dots, e_n(t)$ be an arbitrary fundamental system of solutions of system (1). By the Perron–Vinograd method [7], we reduce system (1) to triangular form

$$dz/dt = B(t)z, \qquad \text{where } B(t) = \begin{pmatrix} b_{11}(t) & 0 \\ \vdots & \ddots \\ \vdots & \ddots \\ b_{n1}(t) & b_{nn}(t) \end{pmatrix}.$$

In this case

$$\|e_1(t)\| = \exp\left[\int_{t_0}^{t} b_{11}(\tau)\, d\tau\right].$$

Now consider the system

$$du_i/dt = b_{ii}(t)u_i \qquad (i = 1, 2, \dots, n). \tag{1'}$$

Denote

$$h_i(t) = \left\{ 0 \ldots \exp \left[\int_{t_0}^{t} b_{ii}(\tau)\, d\tau \right] \ldots 0 \right\}.$$

$$\underbrace{\qquad\qquad}_{\text{in the } i\text{-th place}}$$

Let $t_k \geq t_0$, and suppose that the sequence $\alpha_i\, k h_i(t_k + t)$, for each $i = 1, 2, \ldots, n$, converges uniformly on intervals to some vector-function $h_i^*(t)$. Suppose that the system of solutions $h_1^*(t), \ldots, h_n^*(t)$ of system $(1')$ obtained in this way (limiting when $t_n \to \infty$, and ordinary otherwise) satisfies the following condition. (We denote

$$K_{i,1}(t,\tau) \equiv \frac{\|h_i^*(t)\| \cdot \|h_1^*(\tau)\|}{\|h_i^*(\tau)\| \cdot \|h_1^*(t)\|}.$$

) For each $i = 2, \ldots, n$: either

$$K_{i,1}(t,\tau) \geq C e^{\varepsilon(t-\tau)} > 0$$

for all $t \geq \tau \geq t_0$ and some $\varepsilon > 0$; or

$$K_{i,1}(t,\tau) \leq C e^{-\varepsilon(t-\tau)} \tag{*}$$

for all $t \geq \tau \geq t_0$ and some $\varepsilon > 0$; or

$$C_\varepsilon e^{-\varepsilon(t-\tau)} \leq K_i(t,\tau) \leq C'_\varepsilon e^{\varepsilon(t-\tau)}$$

for all $t \geq \tau \geq t_0$ and every $\varepsilon > 0$.

Then the characteristic exponent of the generalized solution of system (1)

$$e_1^*(t) = \lim_{k\to\infty} \alpha_{1,k} e_1(t_k + t),$$

i.e.

$$\overline{\lim_{t\to+\infty}} \frac{\ln \|e_1^*(t)\|}{t},$$

will be called a **rough exponent of system (1)**.

Definition 1. The set of characteristic exponents of all generalized solutions of system (1) will be called the **spectrum** of system (1).

Definition 2. The set of rough exponents of system (1) will be called the **rough spectrum** of system (1).

This name is justified by the following theorem.

Theorem 2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$g(t) = g_1(t) + g_2(t), \qquad g_1(t) < \delta, \qquad \int_t^\infty g_2(\tau)\, d\tau < \delta \quad (t \geq t_0)$$

and λ belongs to the rough spectrum of system (1), then there exists a generalized solution of system (2) with characteristic exponent $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$.

Proof. In the known way (see ([9], [10])) we reduce the theorem to the case of a diagonal matrix $A(t)$. If some additional restrictions are imposed, then the assertion of the theorem follows from the theorem of R. E. Vinograd (see ([9])). In the general case, the proof of this assertion is carried out using the method of ([14]).

The following theorem shows, in particular, that the rough spectrum is never empty.

Theorem 3. *The special characteristic exponent (see (2)) belongs to the rough spectrum Λ of the system and coincides with* $\sup \Lambda$.

Let us note that, analogously to the special characteristic exponent of Persidskii –Krein, one can define the quantity $\inf \Lambda$.

The proof is carried out with the aid of the following lemma.

Lemma. *Let $p_n(t)$ $(t \geq t_0)$ be continuous functions, $|p_n(t)| \leq P < \infty$.*

Let $t_n \geq \tau_n \geq t_0$, $t_n - \tau_n \to \infty$ and

$$\lambda_n = \frac{1}{t_n - \tau_n} \int_{\tau_n}^{t_n} p_n(\tau)\, d\tau \longrightarrow \lambda.$$

Then there exists a sequence θ_{n_k} such that

$$\int_{\theta_{n_k}}^{\theta_{n_k}+t} p_{n_k}(\tau)\, d\tau \longrightarrow q(t) \geq \lambda t$$

uniformly on each interval $0 \leq t \leq T$.

For the proof of the lemma one must take $\delta_n \to 0$, but so that $\delta_n(t_n - \tau_n) \to \infty$, and then put $\theta_n = \sup$ of those $t \in [\tau_n, t_n]$ for which

$$\int_{\tau_n}^{t} p_n(\tau)\, d\tau \leq (\lambda_n - \delta_n)(t - \tau_n),$$

after which one chooses a subsequence n_k, using Ascoli' s theorem (see ([3]), Ch. X, § 4, théorème 1).

Let us consider systems with slowly varying coefficients. System (1) is so called (the definition belongs to K. P. Persidskii; see, for example, ([2]), p. 105) if for every $\varepsilon > 0$ and T there exists θ such that $\|A(t) - A(t_1)\| < \varepsilon$ for $t_1 > 0$, $|t - t_1| < T$.

Theorem 4. *For a system with slowly varying coefficients, the spectrum coincides with the rough spectrum and coincides with the set of partial limits of $\lambda(t)$ as $t \to +\infty$, where $\lambda(t)$ is an eigenvalue of the matrix $A(t)$.*

Proof. If $\lambda_1 = \lim\limits_{t\to+\infty} \lambda(t)$, then it is easy to find $h_1^*(t), \dots, h_n^*(t)$ such that λ_1 is the characteristic exponent $h_1^*(t)$ and condition $(*)$ is satisfied. On the other hand, it is easy to prove that the spectrum of a system with slowly varying coefficients belongs to the set $\lim\limits_{t\to+\infty} \lambda(t)$, etc. In M. G. Krein (see [(2)], pp. 104–109) it is in fact proved that, for a system with slowly varying coefficients, the spectrum (see Definition 1 above) lies to the left of the point $\sup \lim\limits_{t\to\infty} \lambda(t)$.

Perron' s example (see [(4)]; [(1)], p. 199)

$$\frac{dx}{dt} = (\sin \ln t + \cos \ln t)x,$$

$$\frac{dy}{dt} = (-\sin \ln t + \cos \ln t)y$$

gives a system with slowly varying coefficients ($\|A'(t)\| \to 0,\ t \to \infty$).

Using Theorem 4, the rough spectrum is easily computed: it is the interval $[-\sqrt{2}, +\sqrt{2}]$.

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Note: Figure translations are in progress. See original paper for figures.