

# ZERO-DIMENSIONAL STATIONARY SUBSETS OF FINITE-DIMENSIONAL METRIC SPACES

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## ZERO-DIMENSIONAL STATIONARY SUBSETS OF FINITE-DIMENSIONAL METRIC SPACES

*(Presented by Academician P. S. Aleksandrov on 1 II 1966)*

**Definition.** A subset  $A$  of a metrizable topological space  $(X, T)$  will be called **stationary** if there exists on the set  $X$  a metric  $\rho$  compatible with the topology  $T$  (i.e., a metric satisfying the condition  $T_\rho = T$ , where  $T_\rho$  is the topology induced by the metric  $\rho$ ), and moreover the following condition is fulfilled:

(U). For every continuous mapping  $f : X \rightarrow X$  for which  $\rho(x, f(x)) < 1$  for all points  $x \in X$ , the relation

$$A \cap f(X) = \emptyset$$

holds.

Let us note that if the space  $(X, \rho)$  is a compact metric space, then the set  $A \subset X$  is stationary in the topological space  $(X, T_\rho)$  if and only if there exists a number  $\varepsilon > 0$  such that, for every continuous mapping  $f : X \rightarrow X$  satisfying the condition  $\rho(x, f(x)) < \varepsilon$  for all points  $x \in X$ , one has

$$A \cap f(X) \neq \emptyset.$$

In this note we prove the existence of "small" stationary sets in the sense indicated below.

**Theorem.** *In every finite-dimensional (in the sense of dim) metrizable space  $(X, T)$  there exists a zero-dimensional closed stationary subset.*

**Proof.** Let  $\dim X = n$ , where  $n$  is a natural number (the cases  $n = 1$  and  $n = 0$  are excluded as trivial). Then, as is known, there exist closed sets  $A_i \subset X$ ,  $i = \pm 1, \pm 2, \dots, \pm n$ , such that for any system of closed partitions  $F_i$  between the sets  $A_{-i}$  and  $A_i$ ,  $i = 1, 2, \dots, n$ , we have

$$\bigcap_{i=1}^n F_i \neq \emptyset.$$

But, on the other hand, since the space  $(X, T)$  is metrizable,  $\dim X = \text{Ind } X$ , and therefore there exist such closed partitions  $F_i$  that

$$\dim \bigcap_{i=1}^k F_i = n - k.$$

In particular,

$$\dim \bigcap_{i=1}^n F_i = 0.$$

We shall prove that the set

$$F = \bigcap_{i=1}^n F_i$$

is stationary.

Indeed, let  $\rho_0$  be any metric on the space  $X$ , compatible with the topology  $T$ , and let  $g_i : X \rightarrow I$  be continuous mappings of the space  $X$  into the interval  $I = [-1, 1]$  such that  $g_i(x) = \varepsilon$  for all points  $x \in A_{\varepsilon i}$ ,  $\varepsilon = -1$  or  $1$ , and  $g_i(x) = 0$  for all points  $x \in F_i$ ; finally,  $0 < |g_i(x)| < 1$  for all points  $x \in X \setminus (A_{-i} \cup F_i \cup A_i)$ . Define a new metric on the space  $X$  by the equality

$$\rho(x, y) = \rho_0(x, y) + \max_i |g_i(x) - g_i(y)|.$$

The metric  $\rho$  is also compatible with the topology  $T$ .

Let now  $f : X \rightarrow X$  be an arbitrary continuous mapping, with  $\rho(x, f(x)) < 1$  for all  $x \in X$ . Then  $g_i f(x) < 0 < g_i f(y)$  for all points  $x \in A_{-i}$ ,  $y \in A_i$ ; hence it follows that the sets  $(g_i f)^{-1}(0)$  are partitions between the sets  $A_{-i}$  and  $A_i$ ,  $i = 1, 2, \dots, n$ . Therefore there exists a point  $x \in X$  such that  $g_i f(x) = 0$  for all indices  $i = 1, 2, \dots, n$ . But then  $f(x) \in F$ , i.e.  $F \cap f(x) \neq \emptyset$ . The theorem is proved.

This result can be generalized.

**Definition\*.** A subset  $A$  of a topological space  $(X, T)$  will be called **stationary** if there exists on it a continuous pseudometric  $\rho$  for which condition  $(\bar{y})$  is satisfied.

Then we have

**Theorem\*.** *In every perfectly normal space  $(X, T)$  for which  $\dim X = \text{Ind } X < \infty$ , there exists a closed stationary zero-dimensional subset.*

**Problem.** Does there exist a metrizable space that contains no zero-dimensional closed stationary subset? Is the Hilbert brick  $I^{\aleph_0}$  such a space?

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*Note: Figure translations are in progress. See original paper for figures.*

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