On Effectively \(\Phi\)-Inseparable Sets

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Abstract

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On Effectively Φ-Inseparable Sets

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By analogy with the concepts, due to P. S. Novikov, of an effectively non-enumerable set and of effective distinguishability ([1−3]), in the present article the concept of effective Φ-inseparability of two sets of a metric space is introduced. The properties of effectively Φ-inseparable sets, expressed by Theorems 1-4, are clarified. In the exposition we use the definitions adopted in ([3]). In particular, the symbols $\Pi(E)$, $\Pi_\Phi(E)$, $F(E)$, $\Pi t(E)$ are given the same meaning as in ([3]), pp. 129-130. In addition, for a metric space S we shall denote by $\hat{S}$ the metric product $S \times S$. If, however, S is an abstract set, then $\hat{S}$ will mean the set of pairs (x, y), where $x \in S$, $y \in S$.

1°. **Definition.** Let E be a metric space, P and Q disjoint sets situated in it, and let Φ be a δs-operation. The sets P and Q will be called **effectively Φ-inseparable** if there exist a compact set Z, $0 \subset Z \subseteq E$, a Φ-base $\Pi_\Phi(E)$, and a mapping ν of the set $\widehat{\Pi_\Phi(E)}$ into the set Z, for which: a) whatever the pair $(\{F'_n\}, \{F''_n\}) \in \widehat{\Pi_\Phi(E)}$, one has

$$\nu(\{F'_n\}, \{F''_n\}) \in P \cdot C\Phi\{F'_n\} + Q \cdot C\Phi\{F''_n\} + \Phi\{F'_n\} \cdot \Phi\{F''_n\},$$

b) $\Pi t(Z) \subseteq \Pi_\Phi(E)$; c) the mapping ν is continuous on $\Pi t(Z)$.

In this case ν will be called a **function of effective Φ-inseparability** of P and Q, and the compact set Z a **metric base** of the function ν. We agree to denote the point

$$\nu(\{F'_1, F'_2, \dots, F'_n, \dots\}, \{F''_1, F''_2, \dots, F''_n, \dots\})$$

also by

$$\nu\begin{pmatrix} \{F'_1, & F'_2, & \dots, & F'_n, & \dots\} \\ \{F''_1, & F''_2, & \dots, & F''_n, & \dots\} \end{pmatrix}.$$

We shall call disjoint sets P and Q of the space E (simply) **Φ-inseparable** if for any pair $(\{F'_n\}, \{F''_n\}) \in \widehat{\Pi_\Phi(E)}$

$$P \cdot C\Phi\{F'_n\} + Q \cdot C\Phi\{F''_n\} + \Phi\{F'_n\} \cdot \Phi\{F''_n\} \neq 0.$$

The existence, for every δs-operation Φ, of effectively Φ-inseparable sets in a space E containing a discontinuum follows from the fact that in such a space there exists a set T effectively distinguishable from all Φ-sets of the space E (([3]), p. 146). The disjoint sets $P = T$ and $Q = CT = E - T$ are effectively Φ-inseparable. The Φ-base $\Pi_\Phi(E)$ and the metric base Z of the function ν_Φ of effective distinction for T are then preserved also for the function ν of effective Φ-inseparability of P and Q; the function ν itself is defined by the equality $\nu(\{F'_n\}, \{F''_n\}) = \nu_\Phi\{F'_n\}$, which, for any pair $(\{F'_n\}, \{F''_n\}) \in \widehat{\Pi_\Phi(E)}$, gives:

$$\nu(\{F'_n\}, \{F''_n\}) = \nu_\Phi\{F'_n\} \in P \cdot C\Phi\{F'_n\} + \Phi\{F'_n\} \cdot CP = P \cdot C\Phi\{F'_n\} + \Phi\{F'_n\} \cdot Q$$

$$\subseteq P \cdot C\Phi\{F'_n\} + (C\Phi\{F''_n\} + \Phi\{F'_n\} \cdot \Phi\{F''_n\}) \cdot Q \subseteq P \cdot C\Phi\{F'_n\} + Q \cdot C\Phi\{F''_n\} + \Phi\{F'_n\} \cdot \Phi\{F''_n\}.$$

If the set P is (simply) distinct from all Φ-sets of the space E, then, obviously, the sets P and $Q = CP = E - P$ are (simply) Φ-inseparable in E. For every δs-operation Φ more powerful than the operation of upper limit, a space E in which there exist effectively Φ-in-

divisible sets necessarily contains a discontinuum (see below, items 5°, 6°). At the same time there exist such spaces which contain no discontinuum, but contain sets distinct from all Φ-sets (see ([3]), p. 133). From this follows the existence of a space E with the following property (λ): in E there are Φ-inseparable sets which are not effectively Φ-inseparable. If an arbitrary discontinuum is split, with the aid of the axiom of choice and ordinal numbers, into two completely imperfect sets, then at least one of them, considered as an independent space, has property (λ).

2°. Let us turn to Definition p. 1°. The sets $P \cdot Z$ and $Q \cdot Z$, as is easy to prove (cf. ([3]), p. 136), are effectively Φ-inseparable in the space Z. It is also easy to see the converse: if P and Q are disjoint sets of a space E containing such a compactum Z that $P \cdot Z$ and $Q \cdot Z$ are effectively Φ-inseparable in Z, then P and Q are effectively Φ-inseparable in E.

It follows from this, first, that having one pair of effectively Φ-inseparable sets of the space E, we can form infinitely many other pairs of effectively Φ-inseparable sets in E. Secondly, we obtain the following proposition on the general type of properties of effectively Φ-inseparable sets.

In order that a property α be inherent in every pair of effectively Φ-inseparable sets of the space E, it is necessary and sufficient that

$$\prod_{(P,Q)} \exists_Z \prod_{L \subseteq E - Z} ((P \cdot Z + L;\ Q \cdot Z + L) \text{ has property } \alpha).$$

Here (P, Q) is a pair of effectively Φ-inseparable sets of the space E; Z is a compactum contained in E; $\prod$ and $\exists$ are the quantifiers of universality and existence, respectively (cf. ([4]), p. 438).

From this it is easy to see that measurability and the Baire property do not belong to the general type of properties generated by effective Φ-inseparability.

Properties of this type should be sought among those which pass from the pair $(P \cdot Z;\ Q \cdot Z)$ to the pair $(P \cdot Z + L;\ Q \cdot Z + L)$, where L is any set $\subseteq E - Z$. Some such properties are considered below.

3°. If the δs-operation Φ is stronger than the δs-operation Ψ, and the sets P and Q are effectively Φ-inseparable in the space E, then P and Q are also effectively Ψ-inseparable in E.

The proof of this assertion is analogous to the proof of the corresponding theorem on effective difference in ([5]), p. 297. Reasoning analogous to that proof also leads to the following proposition.

If the operation Φ is stronger than each of the operations Ψ_1 and Ψ_2, and the sets P and Q are effectively Φ-inseparable in the space E, then these sets P and Q possess the following property, which we shall call their **effective** $\Psi_1\Psi_2$**-inseparability**. There exists a compactum Z, $0 \subset Z \subseteq E$, and also such bases $\Pi_{\Psi_1}(E) = \Pi_{\Psi_2}(E)$ and a mapping ν of the set

$$\Pi_{\Psi_1\Psi_2}(E) = \Pi_{\Psi_1}(E) \times \Pi_{\Psi_2}(E)$$

into Z, that: a) for any pair $(\{F'_n\}, \{F''_n\}) \in \Pi_{\Psi_1\Psi_2}(E)$,

$$\nu(\{F'_n\}, \{F''_m\}) \in P \cdot CM' + Q \cdot CM'' + M' \cdot M'',$$

where $M' = \Psi_1\{F'_n\}$, $M'' = \Psi_2\{F''_n\}$; b) $\Pi t(Z) \subseteq \Pi_{\Psi_1}(E) \cdot \Pi_{\Psi_2}(E)$; c) the mapping ν is continuous on $\Pi t(Z)$.

We shall call ν the function of effective $\Psi_1\Psi_2$-inseparability of the sets P and Q, and Z the metric basis of the function ν.

4°. We shall say that the δs-operation Φ has property (ω) if: 1) all its chains are infinite; 2) for every n_0 there is a chain $\{n_1, n_2, ...\}$ of the operation Φ such that $n_0 < n_1 < n_2 < \cdots$; 3) there exists a function χ which puts into one-to-one correspondence with each sequence of sequences of sets $\{S_{m1}\}, \{S_{m2}\}, \ldots, \{S_{mk}\}, \ldots$ a sequence of the same sets $\{S_1, S_2, \ldots, S_n, \ldots\}$, such that

$$\Phi\left(\sum_m S_{m1},\ \sum_m S_{m2},\ \ldots,\ \sum_m S_{mk},\ \ldots\right) = \Phi\{S_1, S_2, \ldots, S_n, \ldots\}. \qquad (*)$$

If S_{mk} are closed sets of the space E and v is a function of effective distinction (for the set T), then by the symbol $v\{\sum_m S_{m1}, \sum_m S_{m2}, \ldots, \sum_m S_{mk}, \ldots\}$ we shall denote the point $v\{S_1, S_2, \ldots, S_n, \ldots\}$ (see equality $(*)$). If, however, v is a function of effective Φ-inseparability (for the sets P and Q of the space E), then, again taking equality $(*)$ into account, by

$$v\left\{\begin{array}{llll} \sum_m S'_{m1}, & \sum_m S'_{m2}, & \ldots, & \sum_m S'_{mk}, \ldots \\ \sum_m S''_{m1}, & \sum_m S''_{m2}, & \ldots, & \sum_m S''_{mk}, \ldots \end{array}\right\}$$

we denote the point

$$v\left\{\begin{matrix} S_1', & S_2', & \dots, & S_n', \dots \\ S_1'', & S_2'', & \dots, & S_n'', \dots \end{matrix}\right\}.$$

Thus, if Φ is an operation with property (ω), then under the sign of a function of effective Φ-inseparability v there may stand sequences of R_σ-sets. Moreover, by virtue of $(*)$, the assertion

$$v(\{F_n'\}, \{F_n''\}) \in P \cdot C\Phi\{F_n'\} + Q \cdot C\Phi\{F_n''\} + \Phi\{F_n'\} \cdot \Phi\{F_n''\}$$

is also preserved for F_σ-sets (in particular, for open sets F_n', F_n''). All this also applies to effective $\Phi_1\Phi_2$-inseparability, if Φ_1 and Φ_2 have property (ω).

Let us note that all δs-operations which precisely produce the B-sets of each given class, beginning with $F_{\sigma\delta}$, as well as A. S. Alexandrov' s A-operation and all more powerful operations, possess property (ω).

5°. By Φ we mean an operation with property (ω).

Theorem 1. *If the sets P and Q of the space E are effectively Φ-inseparable, then each of them contains some discontinuum. Moreover, for any pair of Φ-sets M, N, for which $M \subseteq P$, $N \supseteq Q$, $M \cdot N = 0$, there exists a discontinuum D_P such that $D_P \subset P \cdot (E - M)$. Likewise, for any pair of Φ-sets M, N, for which $M \supseteq P$, $N \subseteq Q$, $M \cdot N = 0$, there exists a discontinuum D_Q such that $D_Q \subset Q \cdot (E - N)$.*

The discontinua D_P and D_Q can be constructed by using the constructions by means of which Theorems 1 and 3 in (4) were proved, and, in addition, item 4° of the present article.

6°. By Φ, Ψ, Ψ_c we mean δs-operations with property (ω).

Theorem 2. *Let Ψ and Ψ_c be mutually complementary δs-operations. If the sets P and Q are effectively $\Psi\Psi_c$-inseparable in the space E, then P is effectively distinct from all Ψ-sets, and Q is effectively distinct from all Ψ_c-sets of the space E.*

Proof. Let v_0 be a function of effective $\Psi\Psi_c$-inseparability of the sets P and Q; Z a metric base of this function; $\Pi_\Psi(E)$ a Ψ-base of the space E. Define the function $v\{F_n\}$ on the set $\Pi_\Psi(E)$ as follows: $v\{F_n\} = v_0(\{F_n\}, \{G_n\})$, where $G_n = E - F_n$ (see in 4° the convention concerning $\sum S_{mk}'$ and $\sum S_{mk}''$). Then for any sequence $\{F_n\} \in \Pi_\Psi(E)$, by virtue of the relations $C\Psi_c\{G_n\} = \Psi\{F_n\}$, $P \cdot Q = 0$, we shall have

$$v\{F_n\} = v_0(\{F_n\}, \{G_n\}) \in P \cdot C\Psi\{F_n\} + Q \cdot C\Psi_c\{G_n\} + \Psi\{F_n\} \cdot \Psi_c\{G_n\} = P \cdot C\Psi\{F_n\} + Q \cdot \Psi\{F_n\} \subseteq P \cdot C\Psi\{F_n\} +$$

This, together with the continuity of the function $v\{F_n\}$ on $\Pi(Z)$, means that the set P is effectively distinct from all Ψ-sets of the space E. The effective distinction of Q from all Ψ_c-sets is proved in the same way.

A consequence of what has been proved is

Theorem 3. *If the sets P and Q are effectively Φ-inseparable in the space E (in particular, if P is effectively distinct from all Φ-sets*

in E, and $Q = E - P$); moreover, if the operation Φ is more powerful than the operation Ψ and the operation Ψ_C complementary to it, then each of the sets P and Q is effectively distinct from all Ψ-sets, and also from all Ψ_C-sets of the space E.

Theorem 3 and the main result of the paper ([6]) lead us to the following proposition.

Theorem 4. *Under the conditions of Theorem 3, the sets P and Q contain $C\Psi$-kernels; i.e., there exist compacta Z_1^P and Z_1^Q such that $Z_1^P \cdot P$, $Z_1^Q \cdot Q$ are $C\Psi$-sets of the spaces Z_1^P, Z_1^Q, respectively, effectively distinct from all Ψ-sets of these spaces.*

Let us note as the most interesting the particular case of Theorems 3 and 4 in which Φ is an A-operation and Ψ is the operation B_α, which yields exactly all B-sets of class α, where α is any ordinal number $< \Omega$. In this case, i.e., when P and Q are effectively A-inseparable, both these sets P and Q contain absolute B_α-sets effectively inseparable, respectively, from $E - P$ and $E - Q$ by any B_β-sets, where $\beta < \alpha$.

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Note: Figure translations are in progress. See original paper for figures.