ON PROPERTIES OF THE SPECTRA OF ERGODIC DYNAMICAL SYSTEMS WITH LOCALLY COMPACT TIME

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ON PROPERTIES OF THE SPECTRA OF ERGODIC DYNAMICAL SYSTEMS WITH LOCALLY COMPACT TIME

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In the work of Ya. G. Sinai ([1]), under a certain assumption called by him condition A, a number of results were obtained which should be regarded as analogues of spectral properties of ergodic dynamical systems with discrete spectrum. Condition A is natural for dynamical systems of probability-theoretic origin. At the same time examples are known in which it is not fulfilled ([2]). Therefore the question is of interest whether the results of Ya. G. Sinai carry over to systems that do not satisfy condition A. It has proved possible to answer this question in the case when the time in the dynamical system is a certain discrete commutative group. The present work therefore consists of two parts: in the first part we carry over the results of ([1]) to the case of dynamical systems with locally compact time; in the second we construct an example of a system not satisfying condition A, and investigate the spectral properties of this system.

1. Let G be a commutative separable locally compact group; (X, μ) a space with normalized measure such that $L^2(X, \mu)$ is separable; $g \to T_g$ a Haar-measurable representation of the group G by automorphisms of the space (X, μ); $g \to U_g$ the unitary representation of G in $L^2(X, \mu)$ conjugate with the representation $g \to T_g$.

The spectral theorem for this case gives

$$U_g = \int_{\hat{G}} \langle g, \chi \rangle E(d\chi), \quad \chi \in \hat{G};$$

where $\hat{G}$ is the character group of the group G; $E(\Delta)$ is a Borel spectral measure on $\hat{G}$, whose values are projection operators in $L^2(X, \mu)$.

Definition. A function $f \in L^2(X, \mu)$ belongs to the class $F^{k,l}$ if, for all integers k_1, l_1, $0 \leq k_1 \leq k$, $0 \leq l_1 \leq l$, there exist complex generalized measures of bounded variation M_{k_1,l_1} on $\hat{G}^{k_1+l_1} = \hat{G} \times \cdots \times \hat{G}$ ($k_1 + l_1$ times) such that

$$M_{k_1,l_1}(\Delta_1 \times \cdots \times \Delta_{k_1} \times \Delta'_1 \times \cdots \times \Delta'_{l_1}) =$$
$$= \int_X E(\Delta_1) f \cdots E(\Delta_{k_1}) f \, E(\Delta'_1) f \cdots E(\Delta'_{l_1}) f \, d\mu;$$

here Δ_i, Δ'_j are Borel subsets of $\hat{G}$.

As in ([3]), we shall study properties of the measure $M_{2,2}$. From the unitarity of U_g it follows that $M_{2,2}(\chi_1, \chi_2; \chi'_1, \chi'_2)$ is concentrated on the subset $\chi_1\chi_2 = \chi'_1\chi'_2$. Moreover, $M_{2,2}(\chi_1, \chi_2; \chi'_1, \chi'_2)$ is symmetric in the variables χ_1, χ_2 and χ'_1, χ'_2.

Let $\psi_\Lambda(\chi, \chi')$ be the characteristic function of a measurable subset $\Lambda \subset G^2$. Define a countably additive vector measure with values–

with values in $L^2(X, \mu)$

$$X_{1,1}(\Lambda) = \int_{\hat{G}^2} \psi_\Lambda(\chi, \chi') \, E(d\chi) f \, E(d\chi') f.$$

Theorem 1. If $\Lambda \subset \{(\chi, \chi') : \chi = \chi'\}$ and the representation $g \to T_g$ is ergodic, then almost everywhere on X, with respect to the measure μ,

$$X_{1,1}(\Lambda) = M_{1,1}(\Lambda).$$

Corollary. Let the representation $g \to T_g$ be ergodic and let $\{\Lambda_k\}$ be a monotonically decreasing sequence of subsets of $\hat{G}^2$ such that

$$\Lambda = \bigcap_k \Lambda_k \subset \{(\chi, \chi) : \chi = \chi'\};$$

then

$$\lim_{k\to\infty} X_{1,1}(\Lambda_k) = M_{1,1}(\Lambda)$$

almost everywhere.

In what follows, in part 1 the ergodicity of the representation $g \to T_g$ is always assumed.

Theorem 2. Let Ξ_1, Ξ_2 be measurable subsets of the diagonal in $\hat{G}^2$, and

$$\Xi = \Xi_1 \times \Xi_2 = \{(\chi_1, \chi_2; \chi'_1, \chi'_2) : (\chi_1, \chi'_1) \in \Xi_1, \ (\chi_2, \chi'_2) \in \Xi_2\};$$

then

$$M_{2,2}(\Xi) = M_{1,1}(\Xi_1) M_{1,1}(\Xi_2). \tag{1}$$

The proof of this theorem differs only slightly from the corresponding proof in (3). It is enough to establish the decomposition (1) for sets $\Xi_i, \ i = 1, 2$, of the form

$$\{(\chi, \chi') : \chi = \chi', \ \chi \in \Delta_i\},$$

where Δ_i are Borel sets. Choose in G a countable everywhere dense subset Π such that, if $g \in \Pi$, then $g^k \in \Pi$. Let

$$\delta_0 = \{e^{i\varphi} : 0 \leq \varphi < \pi\}, \qquad \delta_1 = \{e^{i\varphi} : \pi \leq \varphi < 2\pi\}.$$

Put

$$\Delta_{i_1 \dots i_a} = \{\chi : \chi(g_a) \in \delta_{i_a},\ a = 1, \dots, k\}, \qquad i_k = 0, 1;$$

the $\Delta_{i_1 \dots i_a}$ do not intersect and

$$\bigcup_{i_1 \dots i_k} \Delta_{i_1 \dots i_k} = \hat{G}.$$

Consider

$$K_n^i = \bigcup_{i_1 \dots i_n} (\Delta_i \cap \Delta_{i_1 \dots i_n}) \times (\Delta_i \cap \Delta_{i_1 \dots i_n}) \in \hat{G}^2, \qquad i = 1, 2.$$

It is easy to verify that the sequence K_n^i decreases and

$$\bigcap_n K_n^i = \Xi_i.$$

By the corollary to Theorem 1,

$$M_{2,2}(K_n^1 \times K_m^2) \to M_{1,1}(\Xi_1) \cdot M_{1,1}(K_m^2) \quad \text{as } n \to \infty.$$

Since $M_{2,2}$ is countably additive, we have

$$\lim_{n \to \infty} M_{2,2}(K_n^1 \times K_m^2) = M_{2,2}(\Xi_1 \times K_m^2).$$

Passing to the limit in m, we obtain (1).

Observe that

$$M_{1,1}(\{(\chi, \chi') : \chi = \chi',\ \chi \in \Delta\}) = \sigma_f(\Delta),$$

where

$$\sigma_f(\Delta) = (E(\Delta) f, f).$$

Thus, on the subset $\chi_1 = \chi_1',\ \chi_2 = \chi_2'$ in $\hat{G}^4$, the measure $M_{2,2}(\chi_1, \chi_2; \chi_1', \chi_2')$ reduces to

$$\sigma_f(\chi_1) \cdot \sigma_f(\chi_2).$$

The same is valid for the subset $\chi_1 = \chi_2',\ \chi_2 = \chi_1'$, by virtue of the symmetry of the measure $M_{2,2}$. From Theorem 2 it follows:

Theorem 3. If $f \in F^{2,2}$ and

$$\sigma_f(\Delta) = (E(\Delta) f, f),$$

then there exists a subspace $H \subset L^2(X,\mu)$, invariant with respect to the group U_g, such that the maximal spectral type U of the group U_g in H is subordinated to the type

$$\sigma_f * \sigma_f.$$

The proof differs almost not at all from the proof given in (1) for the case of real time.

Theorem 4. If the vectors from $F^{2,2}$ are everywhere dense in $L^2(X,\mu)$ (condition A), then the maximal spectral type σ of the representation $g \to U_g$ subordinates its convolution

$$\sigma * \sigma.$$

2. We shall construct a representation $g \to T_g$ for which the hypothesis and the conclusion of Theorem 4 do not hold. As G take the discrete group of dyadic-rational numbers (mod 1); (X,μ) is the direct product of the circle Y with Lebesgue measure and the two-point set Z with measures $(1/2, 1/2)$. Let

$$\varepsilon_n = 1/2^n \in G.$$

Put

$$T_{\varepsilon_0} = E$$

—the identity transformation;

$$T_{\varepsilon_k} : (y,i) \to (y + \varepsilon_{k-1}, \alpha(y,\varepsilon_k)i),$$

where $y \in Y,\ i = 1,-1;\ k = 1,2,\ldots.$ We require that the collection of functions $\alpha(y,\varepsilon_k)$, taking the values 1 and -1, satisfy the conditions

$$\alpha(y,\varepsilon_{k+1})\alpha(y+\varepsilon_k,\varepsilon_{k+1}) = \alpha(y,\varepsilon_k), \tag{2}$$

$$\alpha(y,\varepsilon_k) = -1 \quad \text{for } 0 < y < 1/2^{k-1}. \tag{3}$$

Lemma 1. A system of functions satisfying (2), (3) exists.

The transformations T_{ε_k} are automorphisms of the space (X,μ) and

$$T^2_{\varepsilon_{k+1}} = T_{\varepsilon_k}.$$

If

$$g = \sum_k i_k \varepsilon_k, \qquad i_k = 0, 1,$$

then put

$$T_g = \prod_k T^{i_k}_{\varepsilon_k}.$$

The mapping $g \to T_g$ is a measurable representation of G in (X,μ). It is ergodic.

Let H_1 be the subspace of $L^2(X,\mu)$ consisting of functions $f(x) = f(y,i)$ such that $f(y,1) = f(y,-1)$; and let H_{-1} be the subspace of $L^2(X,\mu)$ consisting of functions for which $f(y,1) = -f(y,-1)$. Then

$$L^2(X,\mu) = H_1 \oplus H_{-1}.$$

The spectrum of the group U_g in the invariant subspace H_1 is discrete. The eigenvalues are precisely those characters $\chi \in \widehat{G}$ for which

$$\chi(\varepsilon_k) \to 1 \quad \text{and} \quad \chi(\varepsilon_1) = 1.$$

We shall determine the structure of the spectrum of the group U_g in the invariant subspace H_{-1}. If $y = 0, i_1 \dots i_n, \dots$ is the binary expansion of $y \in Y$, then, by definition, put

$$0 i_1 \dots i_n = \{(y,1) : y = 0, i_1 \dots i_n \dots\},$$
$$1 i_1 \dots i_n = \{(y,-1) : y = 0, \bar{i}_1 \dots \bar{i}_n \dots\},$$

where $\bar{i} = 0$ if $i = 1$, and $\bar{i} = 1$ if $i = 0$.

Let $\{\lambda_n\}$ be a sequence of complex numbers such that

$$|\lambda_n| = 1, \qquad \lambda_0 = 1, \qquad \lambda_{n+1}^2 = \lambda_n.$$

Introduce functions φ_{λ_n} satisfying the conditions

$$U_{\varepsilon_n}\varphi_{\lambda_n} = \lambda_n \varphi_{\lambda_n}, \qquad \varphi_{\lambda_n} = 1 \quad \text{for } x \in 0 \dots 0 \ (n \text{ times}).$$

Suppose that $f \in L^2(X,\mu)$ is a normalized eigenfunction of the group U_g:

$$U_{\varepsilon_n} f = \lambda_n f, \qquad \lambda_{n+1}^2 = \lambda_n, \qquad \lambda_0 = 1, \qquad |f| = 1.$$

Then there exist complex numbers c_n, $|c_n| = 1$, such that

$$c_n \varphi_{\lambda_n} \to f$$

as $n \to \infty$. From the definition of the functions φ_{λ_n} it follows that

$$\|c_{n+1}\varphi_{\lambda_{n+1}} - c_n \varphi_{\lambda_n}\|^2 = \frac{1}{2}|c_{n+1}\lambda_{n+1} - c_n \lambda_1|^2 + \frac{1}{2}|c_{n+1} - c_n|^2.$$

Hence $\lambda_{n+1} \to \lambda_1$, but the sequence $\{\lambda_n\}$ cannot have limit -1. Thus the eigenfunction f must be even. Therefore the spectrum of the group U_g in H_{-1} is continuous.

Let us prove its simplicity. The vector

$$h_1(x) = 1, \quad \text{if } x \in 0; \qquad h_1(x) = -1, \quad \text{if } x \in 1$$

is cyclic in H_{-1}. In general, put

$$h_n(x) = 1, \quad \text{if } x \in 0\dots0 \ (n \text{ times});$$

$$h_n(x) = -1, \quad \text{if } x \in 1\dots1 \ (n \text{ times});$$

$$h_n(x) = 0 \quad \text{at the remaining points.}$$

Suppose that, by linearly combining the shifts $U_g h_1$, one can obtain the function h_n; then in the same way one can also obtain h_{n+1}. Consider the shifts

$$U^k_{\varepsilon_{n+1}} h_n, \qquad k = 0, 1, \dots, 2^n - 1.$$

Note that for $n > 1$

$$\operatorname{supp} h_n \cap \operatorname{supp} U_{\varepsilon_{n+1}} h_n = 0\dots01 \cup 1\dots10 \quad (n+1 \text{ times})$$

and on

$$\underbrace{0\dots01}_{n+1} \cup \underbrace{1\dots10}_{n+1}$$

the functions h_n and $U_{\varepsilon_{n+1}} h_n$ differ in sign;

$$\operatorname{supp} h_n \cap \operatorname{supp} U^{2^n-1}_{\varepsilon_{n+1}} h_n = \underbrace{0\dots0}_{n+1} \cup \underbrace{1\dots1}_{n+1},$$

and on the intersection of the supports the functions h_n and

$$U^{2^n-1}_{\varepsilon_{n+1}} h_n$$

coincide. If, however,

$$1 < k < 2^n - 1,$$

then

$$\operatorname{supp} h_n \cap \operatorname{supp} U^k_{\varepsilon_{n+1}} h_n = \emptyset.$$

Since $T_{\varepsilon_{n+1}}$ is an automorphism, we have

$$\operatorname{supp} U^k_{\varepsilon_{n+1}} h_n \cap \operatorname{supp} U^{k+1}_{\varepsilon_{n+1}} h_n = T^{-k}_{\varepsilon_{n+1}} \underbrace{0\dots01}_{n+1} \cup T^{-k}_{\varepsilon_{n+1}} \underbrace{1\dots10}_{n+1},$$

and on this intersection $U^k_{\varepsilon_{n+1}} h_n$ and $U^{k+1}_{\varepsilon_{n+1}} h_n$ differ in sign, while all other shifts $U^l_{\varepsilon_{n+1}} h_n$ are zero on this set. Hence it follows that, if

$$\sum_{k=0}^{2^n-1} c_k U^k_{\varepsilon_{n+1}} h_n = 0,$$

then

$$c_0 = c_1 = c_2 = \cdots = c_{2^n-2} = c_{2^n-1} = -c_0,$$

and therefore $c_k = 0$. Thus the function h_{n+1} is a linear combination of shifts $U^k_{\varepsilon_{n+1}} h_n$. It remains to note that linear combinations of the functions $U^k_{\varepsilon_n} h_n$ are everywhere dense in H_{-1}.

Put $\sigma_{-1}(\Delta) = (E(\Delta)h_1, h_1)$, $\{\lambda_n\} = \{\chi : \chi(\varepsilon_n) = \lambda_n\}$. Simple calculations give

$$\sigma_{-1}(\{\lambda_n\}) = |(\varphi_{\lambda_n}, h_1)|^2.$$

From the properties of the functions $a(y, \varepsilon_k)$ it follows that

$$(\varphi_{\lambda_n}, h_1) = \frac{1}{2}(1 - \lambda_n)(\varphi^2_{\lambda_n}, h_1).$$

Finally we obtain

$$\sigma_{-1}(\{\lambda_n\}) = \frac{1}{4^n} \prod_{k=1}^{n} |1 - \lambda_k|^2.$$

The sets $\{\lambda_n\}$, taken together, form a net S (for the definition of a net see, for example, (4)). The derivative $d\sigma_{-1}/d\chi$ of the measure σ_{-1} with respect to the net S is represented in the form

$$\lim_{n\to\infty} \frac{1}{2^n} \prod_{k=1}^{n} |1 - \lambda_n|^2.$$

It follows from this representation that $d\sigma_{-1}/d\chi = 0$ almost everywhere, and hence σ_{-1} is singular.

Thus, the spectral characteristic of the representation $g \to T_g$ is as follows: the spectrum is simple; the spectral measure σ is represented in the form $\sigma_1 + \sigma_{-1}$; σ_1 is a discrete measure concentrated on the subgroup of characters χ satisfying the conditions $\chi(\varepsilon_n) \to 1$, $n \to \infty$, $\chi(\varepsilon_1) = 1$; σ_{-1} is a continuous singular measure vanishing on the subgroup $\widehat{G_1}$ of characters for which $\chi(\varepsilon_1) = 1$.

Moreover, $\sigma_{-1} * \sigma_{-1}$ is a continuous measure concentrated on the subgroup $\widehat{G_1}$, and, consequently, σ does not dominate $\sigma * \sigma$.

It follows from this (Theorem 4) that the vectors of the class $F^{2,2}$ do not form an everywhere dense set in $L^2(X, \mu)$. Moreover, if $f \in F^{2,2}$, then (Theorem 3) f necessarily belongs to H_1.

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Note: Figure translations are in progress. See original paper for figures.